Inverse Problems

Local analysis of inverse problems: Hölder stability and iterative reconstruction

Inverse Problems **28** (2012) 045001 (16pp) doi:10.1088/0266-5611/28/4/045001

Local analysis of inverse problems: Hölder stability and iterative reconstruction

Maarten V de Hoop[1], **Lingyun Qiu**[1] **and Otmar Scherzer**[2]

[1] Center for Computational and Applied Mathematics, Purdue University, West Lafayette, IN 47907, USA
[2] Computational Science Center, University of Vienna, Nordbergstraße 15, A-1090 Vienna, Austria and Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria

E-mail: mdehoop@purdue.edu, qiu@purdue.edu and otmar.scherzer@univie.ac.at

Received 13 September 2011, in final form 19 January 2012
Published 6 March 2012
Online at stacks.iop.org/IP/28/045001

Abstract

We consider a class of inverse problems defined by a nonlinear mapping from parameter or model functions to the data, where the inverse mapping is Hölder continuous with respect to appropriate Banach spaces. We analyze a nonlinear Landweber iteration and prove local convergence and convergence rates with respect to an appropriate distance measure. Opposed to the standard analysis of the nonlinear Landweber iteration, we do not assume source and nonlinearity conditions, but this analysis is based solely on the Hölder continuity of the inverse mapping.

1. Introduction

In this paper, we study the convergence of certain nonlinear iterative reconstruction methods for inverse problems in Banach spaces. We consider a class of inverse problems defined by a nonlinear map from parameter or model functions to the data. The parameter functions and data are contained in certain Banach spaces, or Hilbert spaces, respectively. We explicitly construct sequences of parameter functions by a Landweber iteration. Our analysis pertains to obtaining natural conditions for the strong convergence of these sequences (locally) to the solutions in an appropriate distance measure.

Our main result establishes convergence of the Landweber iteration if the inverse problem ensures a Hölder stability estimate. Moreover, we prove monotonicity of the residuals defined by the sequence induced by the iteration. We also obtain the convergence rates without so-called source and nonlinearity conditions. The stability condition is a natural one in the framework of iterative reconstruction.

Extensive research has been carried out to study the convergence of the Landweber iteration [25] and its modifications. In the case of model and data spaces being Hilbert, see [18]. An overview of iterative methods for inverse problems in Hilbert spaces can be found, for example, in [21]. Schöpfer *et al* presented a nonlinear extension of the Landweber method to Banach spaces using duality mappings [30]. We use this iterative method in the analysis presented here. Duality mappings also play a role in iterative schemes for monotone and accretive operators (see [2, 14, 36, 37]). The model space needs to be smooth and uniformly convex; however, the data space can be an arbitrary Banach space. Due to the geometrical characteristics of Banach spaces other than Hilbert spaces, it is more appropriate to use Bregman distances rather than Ljapunov functionals to prove convergence [27]. For convergence rates, see [20]. Schöpfer *et al* furthermore considered the solution of convex split feasibility problems in Banach spaces by cyclic projections [31]. Under the so-called tangential cone condition, pertaining to the nonlinear map modeling the data, convergence has been established, invoking a source condition in a convergence rate result. Here, we build on the work of Kaltenbacher *et al* and revisit these conditions with a view to stability properties of the inverse problem [22].

In many inverse problems one probes a medium, or an obstacle, with a particular type of field and measures the response. From these measurements one aims to determine the medium properties and/or (geometrical) structure. Typically, the physical phenomenon is modeled by partial differential equations and the medium properties by variable, and possibly singular, coefficients. The interaction of fields is usually restricted to a bounded domain with boundary. Experiments can be carried out on the boundary. The goal is thus to infer information on the coefficients in the interior of the domain from the associated boundary measurements. The map, solving the partial differential equations, from coefficients or parameter functions to the measurements or data is nonlinear. Its injectivity is studied in the analysis of inverse problems. As an example, we discuss electrical impedance tomography (EIT), where the Dirichlet-to-Neumann map represents the data, and summarize the conditions leading to Lipschitz stability.

Traditionally, the Landweber iteration has been viewed as a fixed-point iteration. However, in general, for inverse problems, the underlying fixed point operator is not a contraction. There is an extensive literature of iterative methods for approximating fixed points of non-expansive operators. Hanke *et al* replace the condition of non-expansive to a local tangential cone condition, which guarantees a local result [18]. In the finite-dimensional setting, in which, for example, the model space is $\mathbb{R}^n$, non-convex constraint optimization problems admitting iterative solutions have been studied by Curtis *et al* [16]. Under certain assumptions, they obtain convergence to stationary points of the associated feasibility problem. In the context of inverse problems defined by partial differential equations, this setting is motivated by discretizing the problems prior to studying the convergence (locally) of the iterations. Inequality constraints are necessary to enforce locality. The non-convexity is addressed by Hessian modifications based on inertia tests.

This paper is organized as follows. In the following section, we summarize certain geometrical aspects of Banach spaces, including (uniform) smoothness and (uniform) convexity, and their connection to duality mappings. Smoothness is naturally related to Gâteaux differentiability. We also introduce the Bregman distance. We then define the nonlinear Landweber iteration in Banach spaces. In section 3, we introduce the basic assumptions including Hölder stability and analyze the convergence of the Landweber iteration in Hilbert spaces. In section 4, we adapt these assumptions and generalize the analysis of convergence of the Landweber iteration to Banach spaces. We also establish the convergence rates. In section 5, we give an example, namely the reconstruction of conductivity in EIT, and show that our assumptions can be satisfied.

2. Landweber iteration in Banach spaces

Let X and Y both be real Banach spaces. We consider the nonlinear operator equation

$$F(x) = y, \quad x \in \mathcal{D}(F),\ y \in Y, \tag{2.1}$$

with the domain $\mathcal{D}(F) \subset X$. In applications, $F : \mathcal{D}(F) \to Y$ models the data. In the inverse problem one is concerned with the question whether y determines x. We assume that F is continuous and that F is Fréchet differentiable, locally.

We couple the uniqueness and stability analysis of the inverse problem to a local solution construction based on the Landweber iteration. Throughout this paper, we assume that the data y in (2.1) are attainable, that is, that (2.1) has a solution $x^\dagger$ (which needs not to be unique).

2.1. Duality mappings

The duals of X and Y are denoted by X^* and Y^*, respectively. Their norms are denoted by $\|\cdot\|$. We denote the space of continuous linear operators $X \to Y$ by $\mathcal{L}(X, Y)$. Let $A : \mathcal{D}(A) \subset X \to Y$ be continuous. Here, $\mathcal{D}(A)$ denotes the domain of A. Let $h \in \mathcal{D}(A)$ and $k \in X$ and assume that $h + t(k - h) \in \mathcal{D}(A)$ for all $t \in (0, t_0)$ for some t_0; then we denote by $DA(h)(k)$ the directional derivative of A at $h \in \mathcal{D}(A)$ in the direction $k \in \mathcal{D}(A)$, that is,

$$DA(h)(k) := \lim_{t\to 0^+} \frac{A(h + tk) - A(h)}{t}.$$

If $DA(h) \in \mathcal{L}(X, Y)$, then $DA(h)$ is called Gâteaux differentiable. If, in addition, the convergence is uniform for all $k \in B_{t_0}$, then DA is Fréchet differentiable at h. For $x \in X$ and $x^* \in X^*$, we write the dual pair as $\langle x, x^*\rangle = x^*(x)$. We write A^* for the dual operator $A^* \in \mathcal{L}(Y^*, X^*)$ and $\|A\| = \|A^*\|$ for the operator norm of A. We let $1 < p, q < \infty$ be conjugate exponents, that is,

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{2.2}$$

For $p > 1$, the subdifferential mapping $J_p = \partial f_p : X \to 2^{X^*}$ of the convex functional $f_p : x \mapsto \frac{1}{p}\|x\|^p$ defined by

$$J_p(x) = \{x^* \in X^* \mid \langle x, x^*\rangle = \|x\|\,\|x^*\| \text{ and } \|x^*\| = \|x\|^{p-1}\} \tag{2.3}$$

is called the duality mapping of X with the gauge function $t \mapsto t^{p-1}$. Generally, the duality mapping is set-valued. In order to let J_p be single-valued, we need to introduce the notion of convexity and smoothness of Banach spaces.

One defines the convexity modulus δ_X of X by

$$\delta_X(\epsilon) = \inf_{x,\tilde{x}\in X}\{1 - \|\tfrac{1}{2}(x + \tilde{x})\| \mid \|x\| = \|\tilde{x}\| = 1 \text{ and } \|x - \tilde{x}\| \geqslant \epsilon\} \tag{2.4}$$

and the smoothness modulus ρ_X of X by

$$\rho_X(\tau) = \sup_{x,\tilde{x}\in X}\{\tfrac{1}{2}(\|x + \tau\tilde{x}\| + \|x - \tau\tilde{x}\| - 2) \mid \|x\| = \|\tilde{x}\| = 1\}. \tag{2.5}$$

Definition 2.1. *A Banach space X is said to be*

(a) uniformly convex if $\delta_X(\epsilon) > 0$ *for any* $\epsilon \in (0, 2]$,
(b) uniformly smooth if $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 0$,
(c) convex of power type p or p-convex if there exists a constant $C > 0$ *such that* $\delta_X(\epsilon) \geqslant C\epsilon^p$,
(d) smooth of power type q- or q-smooth if there exists a constant $C > 0$ *such that* $\rho_X(\tau) \leqslant C\tau^q$.

Example 2.2.

(a) A Hilbert space X is 2-convex and 2-smooth and $J_2 : X \to X$ is the identity mapping.
(b) Let $\Omega \subset \mathbb{R}^n$ be an open domain. The Banach space $L^p = L^p(\Omega)$, $p > 1$ is uniformly convex and uniformly smooth, and

$$\delta_{L^p}(\epsilon) \simeq \begin{cases} \epsilon^2, & 1 < p < 2, \\ \epsilon^p, & 2 \leqslant p < \infty; \end{cases}$$

$$\rho_{L^p}(\tau) \simeq \begin{cases} \tau^p, & 1 < p < 2, \\ \tau^2, & 2 \leqslant p < \infty. \end{cases}$$

(c) For $X = L^r(\mathbb{R}^n)$, $r > 1$, we have

$$J_p : L^r(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$$
$$u(x) \mapsto \|u\|_{L^r}^{p-r} |u(x)|^{r-2} u(x),$$

where $\frac{1}{r} + \frac{1}{s} = 1$.

For a detailed introduction to the geometry of Banach spaces and the duality mapping, we refer to [15, 30]. We list the properties we need here in the following theorem.

Theorem 2.3. *Let $p > 1$. The following statements hold true.*

(a) For every $x \in X$, the set $J_p(x)$ is not empty and it is convex and weakly closed in X^*.
(b) If a Banach space is uniformly convex, it is reflexive.
(c) A Banach space X is uniformly convex (resp. uniformly smooth) iff X^* is uniformly smooth (resp. uniformly convex).
(d) If a Banach space X is uniformly smooth, $J_p(x)$ is single-valued for all $x \in X$.
(e) If a Banach space X is uniformly smooth and uniformly convex, $J_p(x)$ is bijective and its inverse $J_p^{-1} : X^* \to X$ is given by $J_p^{-1} = J_q^*$, with J_q^* being the duality mapping of X^* with the gauge function $t \mapsto t^{q-1}$, where $1 < p, q < \infty$ are conjugate exponents.

Throughout this paper, we assume that X is p-convex and q-smooth with $p, q > 1$; hence, it is uniformly smooth and uniformly convex. Furthermore, X is reflexive and its dual X^* has the same properties. Y is allowed to be an arbitrary Banach space; j_p will be a single-valued selection of the possibly set-valued duality mapping of Y with the gauge function $t \mapsto t^{p-1}$, $p > 1$. Further restrictions on X and Y will be indicated in the respective theorems below.

2.2. Bregman distances

Due to the geometrical characteristics of Banach spaces different from those of Hilbert spaces, it is often more appropriate to use the Bregman distance instead of the conventional norm-based functionals $\|x - \tilde{x}\|^p$ or $\|J_p(x) - J_p(\tilde{x})\|^p$ for convergence analysis. This idea goes back to [11].

Definition 2.4. *Let X be a uniformly smooth Banach space and $p > 1$. The Bregman distance $\Delta_p(x, \cdot)$ of the convex functional $x \mapsto \frac{1}{p}\|x\|^p$ at $x \in X$ is defined as*

$$\Delta_p(x, \tilde{x}) = \frac{1}{p}\|\tilde{x}\|^p - \frac{1}{p}\|x\|^p - \langle J_p(x), \tilde{x} - x\rangle, \quad \tilde{x} \in X, \tag{2.6}$$

where J_p denotes the duality mapping of X with the gauge function $t \mapsto t^{p-1}$.

In the following theorem, we summarize some facts concerning the Bregman distance and the relationship between Bregman distance and the norm [1, 2, 12, 35].

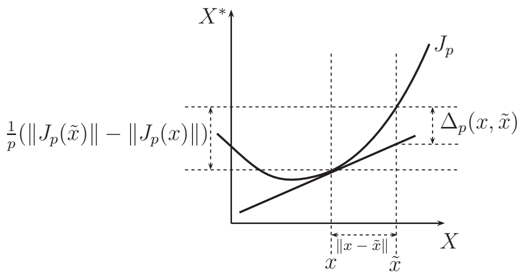

Figure 1. Bregman distance Δ_p.

Theorem 2.5. *Let X be a uniformly smooth and uniformly convex Banach space. Then, for all $x, \tilde{x} \in X$, the following holds:*

(a)

$$\begin{aligned}\Delta_p(x,\tilde{x}) &= \frac{1}{p}\|\tilde{x}\|^p - \frac{1}{p}\|x\|^p - \langle J_p(x), \tilde{x}\rangle + \|x\|^p \\ &= \frac{1}{p}\|\tilde{x}\|^p + \frac{1}{q}\|x\|^p - \langle J_p(x), \tilde{x}\rangle. \end{aligned} \tag{2.7}$$

(b) $\Delta_p(x,\tilde{x}) \geqslant 0$ *and* $\Delta_p(x,\tilde{x}) = 0 \Leftrightarrow x = \tilde{x}$.
(c) Δ_p *is continuous in both arguments.*
(d) The following statements are equivalent:

(i) $\lim_{n\to\infty} \|x_n - x\| = 0$,
(ii) $\lim_{n\to\infty} \Delta_p(x_n, x) = 0$ and
(iii) $\lim_{n\to\infty} \|x_n\| = \|x\|$ *and* $\lim_{n\to\infty}\langle J_p(x_n), x\rangle = \langle J_p(x), x\rangle$.

(e) If X is p-convex, there exists a constant $C_p > 0$ *such that*

$$\Delta_p(x,\tilde{x}) \geqslant \frac{C_p}{p}\|x-\tilde{x}\|^p. \tag{2.8}$$

(f) If X^* *is q-smooth, there exists a constant* $G_q > 0$ *such that*

$$\Delta_q(x^*,\tilde{x}^*) \leqslant \frac{G_q}{q}\|x^* - \tilde{x}^*\|^q, \tag{2.9}$$

for all $x^*, \tilde{x}^* \in X^*$.

Remark 2.6. The Bregman distance Δ_p is similar to a metric, but, in general, does not satisfy the triangle inequality nor symmetry. In a Hilbert space, $\Delta_2(x,\tilde{x}) = \frac{1}{2}\|x-\tilde{x}\|^2$.

2.3. Landweber iteration

In this subsection, we introduce an iterative method for minimizing the functional

$$\Phi(x) = \frac{1}{p}\|F(x) - y\|^p. \tag{2.10}$$

The iterates $\{x_k\}$ are generated with the steepest descent flow given by

$$\partial\Phi^{(k)}(x_k) = DF(x_k)^* j_p(F(x_k) - y). \tag{2.11}$$

To be more precise, we study the iterative method in Banach spaces,

$$J_p(x_{k+1}) = J_p(x_k) - \mu DF(x_k)^* j_p(F(x_k) - y),$$
$$x_{k+1} = J_q^*(J_p(x_{k+1})), \tag{2.12}$$

where $J_p : X \to X^*, J_q^* : X^* \to X$ and $j_p : Y \to Y^*$ denote duality mappings in corresponding spaces. When X and Y are Hilbert spaces and $p = 2$, this reduces to the Landweber iteration in Hilbert spaces

$$x_{k+1} = x_k - \mu DF(x_k)^*(F(x_k) - y). \tag{2.13}$$

If F is a linear operator, the iteration (1.3) coincides with Landweber's original algorithm. We specify μ below. Equation (2.12) defines a sequence (x_k).

If $F(x^\dagger) = y$, the so-called tangential cone condition [22],

$$\|F(x) - F(\tilde{x}) - DF(x)(x - \tilde{x})\| \leqslant c_{\mathrm{TC}}\, \|F(x) - F(\tilde{x})\| \quad \forall\, x, \tilde{x} \in \mathcal{B}_\rho^\Delta(x^\dagger), \tag{2.14}$$

for some $0 < c_{\mathrm{TC}} < 1$, is crucial to obtain the convergence of (x_k) to $x^\dagger$ [19, 20, 22]; $\mathcal{B}_\rho^\Delta(x^\dagger) = \{x \in X \mid \Delta_p(x, x^\dagger) \leqslant \rho\} \subset \mathcal{D}(F)$. A source condition controls the convergence rate. Here, we study convergence and convergence rates in relation to a single, alternative condition replacing the tangential cone and source conditions, namely Hölder-type stability,

$$\Delta_p(x, \tilde{x}) \leqslant C_F^p \|F(x) - F(\tilde{x})\|^{\frac{1+\varepsilon}{2} p} \quad \forall\, x, \tilde{x} \in \mathcal{B}_\rho^\Delta(x^\dagger),$$

for some $\varepsilon \in (0, 1]$. With the Fréchet differentiability of F and the Lipschitz continuity of DF, this condition implies the tangential cone condition, and, hence, convergence is guaranteed; however, it also implies a certain convergence rate.

3. Convergence rate and radius of convergence—Hilbert spaces

In this section, we assume that X and Y are Hilbert spaces. Then the mappings J_p, j_p and J_q^* are identity mappings. Let $\mathcal{B}_\rho(x_0)$ denote a closed ball centered at x_0 with radius ρ, such that $\mathcal{B} = \mathcal{B}_{\rho'}(x_0) \subset \mathcal{D}(F)$, $\rho' > \rho$. As before, let $x^\dagger$ generate the data y, that is,

$$F(x^\dagger) = y. \tag{3.1}$$

We assume that $x^\dagger \in \mathcal{B}_\rho(x_0)$.

Assumption 3.1.

(a) The Fréchet derivative, DF, of F is Lipschitz continuous locally in $\mathcal{B}$ and

$$\|DF(x) - DF(\tilde{x})\| \leqslant L\|x - \tilde{x}\| \quad \forall x, \tilde{x} \in \mathcal{B}. \tag{3.2}$$

(b) F is weakly sequentially closed, that is,

$$\left.\begin{array}{l} x_n \rightharpoonup x, \\ F(x_n) \to y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x \in \mathcal{D}(F), \\ F(x) = y. \end{array}\right.$$

(c) The inversion has the uniform Hölder-type stability, that is, there exists a constant, $C_F > 0$, such that

$$\frac{1}{\sqrt{2}}\|x - \tilde{x}\| \leqslant C_F \|F(x) - F(\tilde{x})\|^{\frac{1+\varepsilon}{2}} \quad \forall\, x, \tilde{x} \in \mathcal{B} \tag{3.3}$$

for some $\varepsilon \in (0, 1]$

In the remainder of this section, we discuss the convergence criterion and convergence rate for the Landweber iteration (2.13).

Theorem 3.2. *Assume there exists a solution* $x^\dagger$ *to (3.1) and that assumption 3.1 holds. Furthermore, assume that*

$$\|DF(x)\| \leqslant \hat{L} \quad \forall x \in \mathcal{B}. \tag{3.4}$$

Let the positive stepsize μ *be such that*

$$\mu < \frac{1}{\hat{L}^2}, \qquad \mu(1-\mu\hat{L}^2) < 2^{\frac{2}{1+\varepsilon}} C_F^{\frac{4}{1+\varepsilon}}. \tag{3.5}$$

Let

$$\rho = \tfrac{1}{2}\big(2L\hat{L}^\varepsilon C_F^2\big)^{-2/\varepsilon}.$$

If

$$\tfrac{1}{2}\|x_0 - x^\dagger\|^2 \leqslant \rho, \tag{3.6}$$

then the iterates satisfy

$$\tfrac{1}{2}\|x_k - x^\dagger\|^2 \leqslant \rho, \qquad k = 1, 2, \ldots \tag{3.7}$$

and $x_k \to x^\dagger$ *as* $k \to \infty$. *Moreover, let*

$$c = \tfrac{1}{2}\mu(1-\mu\hat{L}^2)C_F^{-\frac{4}{1+\varepsilon}}. \tag{3.8}$$

From (3.5), it follows that $0 < c < 1$. *The convergence rate is given by*

$$\tfrac{1}{2}\|x_k - x^\dagger\|^2 \leqslant \rho(1-c)^k, \tag{3.9}$$

if $\varepsilon = 1$. *For* $\varepsilon \in (0,1)$, *the convergence rate is given by*

$$\frac{1}{2}\|x_k - x^\dagger\|^2 \leqslant \left(ck\frac{1-\varepsilon}{1+\varepsilon} + \rho^{-\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \qquad k = 0, 1, \ldots. \tag{3.10}$$

The proof is a special case of the Banach space setting, cf theorem 4.5; see section 4. The convergence is sublinear if $0 < \varepsilon < 1$ and the speed up as $\varepsilon \to 1$ relates to the fact that it switches to a linear convergence.

For the critical index $\varepsilon = 0$, that is, the power on the right-hand side of the stability inequality (3.3) equals to 1/2, we need to invoke an assumption on the stability constant C_F to arrive at the convergence and convergence rate results. An interesting by-product is that the convergence radius only depends on the radius within which the Hölder stability (3.3) holds. Hence, if the forward operator F satisfies (3.3) globally, then we obtain a global convergence and convergence rate result.

Theorem 3.3. *Assume there exists a solution* $x^\dagger$ *to (3.1) and that assumption 3.1 holds with* $\varepsilon = 0$. *Furthermore, assume that*

$$\|DF(x)\| \leqslant \hat{L} \quad \forall x \in \mathcal{B}. \tag{3.11}$$

Let the stability constant C_F *and the positive stepsize* μ *satisfy that*

$$\mu\hat{L}^2 + 2LC_F^2 < 2. \tag{3.12}$$

Then the iterates satisfy

$$x_k \to x^\dagger \text{ as } k \to \infty.$$

Moreover, let

$$c = \frac{\mu}{4}\big(-2 + \mu\hat{L}^2 + 2LC_F^2\big)C_F^{-4}. \tag{3.13}$$

The convergence rate is given by

$$\tfrac{1}{2}\|x_k - x^\dagger\|^2 \leqslant (2\|x_0 - x^\dagger\|^{-2} + ck)^{-1}. \tag{3.14}$$

The proof is again a special case of the Banach space setting, cf theorem 4.5; see section 4.

Remark 3.4. The convergence radius condition (3.6) on the starting point x_0 may be replaced by a convergence radius condition on the starting simulated data $F(x_0)$,

$$\|F(x_0) - y\|^{1+\varepsilon} \leqslant \rho C_F^{-2}. \tag{3.15}$$

In fact, with the aid of the stability inequality (3.3), (3.6) follows from (3.15).

4. Convergence rate and radius of convergence—Banach spaces

In this section, we discuss the convergence and convergence rate of the Landweber iteration (2.12) in Banach spaces. Let $\mathcal{B}_\rho(x_0)$ denote a closed ball centered at x_0 with radius ρ, and $\mathcal{B} = \mathcal{B}_\rho^\Delta(x^\dagger)$ denote a ball with respect to the Bregman distance centered at some solution $x^\dagger$. We assume that $\mathcal{B}_\rho^\Delta(x^\dagger) \subset \mathcal{D}(F)$.

Assumption 4.1.

(a) The Fréchet derivative, DF, of F is Lipschitz continuous locally in $\mathcal{B}$ *and*

$$\|DF(x) - DF(\tilde{x})\| \leqslant L\|x - \tilde{x}\| \quad \forall x, \tilde{x} \in \mathcal{B}. \tag{4.1}$$

(b) F is weakly sequentially closed, that is,

$$\left.\begin{matrix} x_n \rightharpoonup x, \\ F(x_n) \to y \end{matrix}\right\} \Rightarrow \begin{cases} x \in \mathcal{D}(F), \\ F(x) = y. \end{cases}$$

(c) The inversion has the uniform Hölder-type stability, that is, there exists a constant $C_F > 0$ *such that*

$$\Delta_p(x, \tilde{x}) \leqslant C_F^p \|F(x) - F(\tilde{x})\|^{\frac{1+\varepsilon}{2}p} \quad \forall x, \tilde{x} \in \mathcal{B}, \tag{4.2}$$

for some $\varepsilon \in (0, 1]$.

Remark 4.2. Note that the nonemptiness of the interior (with respect to norm) of $\mathcal{D}(F)$ is sufficient for $\mathcal{B} \subset \mathcal{D}(F)$.

Remark 4.3. With the assumption that X is p-convex, (4.2) with (2.8) implies the regular notion of Hölder stability in norm.

Remark 4.4. Under the Lipschitz-type stability assumption, that is, (4.2) with $\varepsilon = 1$, we have that

$$\begin{aligned}\langle J_p(x^\dagger), x - x^\dagger\rangle &\leqslant \|x^\dagger\|^{p-1}\|x - x^\dagger\| \\ &\leqslant C\Delta_p(x, x^\dagger)^{1/p} \\ &\leqslant CC_F\|F(x) - F(x^\dagger)\|, \quad \forall\, x \in \mathcal{B}\end{aligned}$$

for some constant $C > 0$. It has been shown in [29] that this implies the source condition,

$$J_p(x^\dagger) = DF(x^\dagger)^*\omega$$

for some ω satisfying $\|\omega\| \leqslant 1$.

Theorem 4.5. *Let Y be a general Banach space, and X be a Banach space which is p-convex and q-smooth with $1/p + 1/q = 1$. Assume there exists a solution $x^\dagger$ to (3.1) and that assumption 4.1 holds. Furthermore, assume that*

$$\|DF(x)\| \leqslant \hat{L} \quad \forall x \in \mathcal{B}. \tag{4.3}$$

Let the positive stepsize, μ, be such that

$$\begin{aligned} &\mu^{q-1} < \frac{q}{2G_q\hat{L}^q}, \\ &\mu\left(\frac{1}{2} - \frac{G_q\hat{L}^q}{q}\mu^{q-1}\right) < C_F^{\frac{2p}{1+\varepsilon}}. \end{aligned} \tag{4.4}$$

Let

$$\rho = \hat{L}^{-p}(LC_F^2)^{-\frac{p}{\varepsilon}}\left(\frac{C_p}{p}\right)^{1+\frac{2}{\varepsilon}}.$$

If

$$\Delta_p(x_0, x^\dagger) \leqslant \rho, \tag{4.5}$$

then the iterates satisfy

$$\Delta_p(x_k, x^\dagger) \leqslant \rho, \quad k = 1, 2, \ldots \tag{4.6}$$

and $\Delta_p(x_k, x^\dagger) \to 0$ as $k \to \infty$. Moreover, let

$$c = C_F^{-\frac{2p}{1+\varepsilon}}\left(\frac{1}{2}\mu - \frac{G_q}{q}\mu^q\hat{L}^q\right). \tag{4.7}$$

The convergence rate is given by

$$\Delta_p(x_k, x^\dagger) \leqslant \rho(1-c)^k, \tag{4.8}$$

if $\varepsilon = 1$. For $\varepsilon \in (0, 1)$, the convergence rate is given by

$$\Delta_p(x_k, x^\dagger) \leqslant \left(ck\frac{1-\varepsilon}{1+\varepsilon} + \rho^{-\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \quad k = 0, 1, \ldots. \tag{4.9}$$

Proof. Using (2.7) and (2.3), we obtain, for the sequence of residues,

$$\begin{aligned} \Delta_p(x_{k+1}, x^\dagger) &= \Delta_p(x_k, x^\dagger) + \frac{1}{q}(\|x_{k+1}\|^p - \|x_k\|^p) - \langle J_p(x_{k+1}) - J_p(x_k), x^\dagger\rangle \\ &= \Delta_p(x_k, x^\dagger) + \frac{1}{q}(\|J_p(x_{k+1})\|^q - \|J_p(x_k)\|^q) - \langle J_p(x_{k+1}) - J_p(x_k), x^\dagger\rangle. \end{aligned} \tag{4.10}$$

Applying (2.7) and (f) of theorem 2.5 with $x^* = J_p(x_{k+1})$ and $\tilde{x}^* = J_p(x_k)$, we obtain

$$\frac{1}{q}(\|J_p(x_{k+1})\|^q - \|J_p(x_k)\|^q) \leqslant \frac{G_q}{q}\|J_p(x_{k+1}) - J_p(x_k)\|^q + \langle J_p(x_{k+1}) - J_p(x_k), x_k\rangle. \tag{4.11}$$

Substituting (2.12) and using this inequality in (4.10) yield

$$\begin{aligned} \Delta_p(x_{k+1}, x^\dagger) - \Delta_p(x_k, x^\dagger) \leqslant &\frac{G_q}{q}\|\mu DF(x_k)^* j_p(F(x_k) - y)\|^q \\ &- \langle \mu DF(x_k)^* j_p(F(x_k) - y), x_k - x^\dagger\rangle. \end{aligned} \tag{4.12}$$

We estimate each term in (4.12) separately. The first term satisfies the estimate

$$\frac{G_q}{q}\|\mu DF(x_k)^* j_p(F(x_k)-y)\|^q \leqslant \frac{G_q}{q}\mu^q \hat{L}^q \|(F(x_k)-y)\|^p. \tag{4.13}$$

For the second term, we have that

$$\begin{aligned} -\langle \mu DF(x_k)^* j_p(F(x_k)-y), x_k - x^\dagger\rangle &= -\mu\langle j_p(F(x_k)-y), DF(x_k)(x_k-x^\dagger)\rangle \\ &= -\mu(\langle j_p(F(x_k)-y), F(x_k)-y\rangle \\ &\quad - \langle j_p(F(x_k)-y), F(x_k)-y-DF(x_k)(x_k-x^\dagger)\rangle). \end{aligned} \tag{4.14}$$

Note that, by the fundamental theorem of calculus for the Fréchet derivative, we obtain that

$$\|F(x_k)-y-DF(x_k)(x_k-x^\dagger)\| \leqslant \frac{L}{2}\|x_k-x^\dagger\|^2. \tag{4.15}$$

Then, using (2.8) and stability (c) of assumption 4.1, we have

$$\begin{aligned} &-\langle \mu DF(x_k)^* j_p(F(x_k)-y), x_k-x^\dagger\rangle \\ &\quad = -\mu\|F(x_k)-y\|^p + \mu\langle j_p(F(x_k)-y), F(x_k)-y-DF(x_k)(x_k-x^\dagger)\rangle \\ &\quad \leqslant -\mu\|F(x_k)-y\|^p + \frac{\mu}{2}L\|(F(x_k)-y)\|^{p-1}\|x_k-x^\dagger\|^2 \\ &\quad \leqslant -\mu\|F(x_k)-y\|^p + \frac{\mu}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\|F(x_k)-y\|^{p+\varepsilon}. \end{aligned} \tag{4.16}$$

Combining these estimates and using the notation

$$\gamma_k = \Delta_p(x_k, x^\dagger),$$

we obtain

$$\begin{aligned} \gamma_{k+1}-\gamma_k \leqslant &\left(\frac{G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu\right)\|F(x_k)-y\|^p \\ &-\frac{1}{2}\mu\|F(x_k)-y\|^p + \frac{\mu}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\|F(x_k)-y\|^{p+\varepsilon}. \end{aligned} \tag{4.17}$$

We claim that

$$\gamma_{k+1} = \Delta_p(x_{k+1}, x^\dagger) \leqslant \rho, \tag{4.18}$$

which we prove by induction. Assume that

$$\Delta_p(x_m, x^\dagger) \leqslant \rho \tag{4.19}$$

holds for $m = 0, 1, \ldots, k$. With the mean value inequality, it follows that

$$\|F(x_m)-y\|^\varepsilon \leqslant \hat{L}^\varepsilon\left(\frac{p}{C_p}\rho\right)^{\frac{\varepsilon}{p}} = \frac{1}{LC_F^2(p/C_p)^{2/p}}, \qquad m = 0, 1, 2, \ldots, k. \tag{4.20}$$

Therefore,

$$-\tfrac{1}{2}\mu\|F(x_m)-y\|^p + \tfrac{1}{2}\mu LC_F^2(p/C_p)^{2/p}\|F(x_m)-y\|^{p+\varepsilon} \leqslant 0, \tag{4.21}$$

$m = 0, 1, 2, \ldots, k$. Dropping this non-positive term, we obtain

$$\gamma_{k+1}-\gamma_k \leqslant \left(\frac{G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu\right)\|F(x_k)-y\|^p. \tag{4.22}$$

Note that the term $\left(\frac{G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu\right)\|F(x_k)-y\|^p$ is non-positive. We obtain that

$$\Delta_p(x_{k+1}, x^\dagger) \leqslant \rho, \tag{4.23}$$

which establishes the claim.

Now, we return to (4.22). By the Hölder-type stability (4.2), we have that

$$\gamma_{k+1} \leqslant \gamma_k - c\gamma_k^{\frac{2}{1+\varepsilon}} \tag{4.24}$$

Note that, by the conditions on μ, we have $0 < c$. By letting k go to infinity on both sides of the above inequality, we conclude that

$$\gamma_k \to 0 \quad \text{as} \quad k \to \infty.$$

In the remainder of the proof, we obtain the convergence rate. Note that, with the choice (4.4) of μ,

$$0 < c < 1. \tag{4.25}$$

With $\varepsilon = 1$, we have

$$\gamma_{k+1} \leqslant (1 - c)\gamma_k \tag{4.26}$$

which expresses the convergence rate (4.8).

For the convergence rate with $\varepsilon \in (0, 1)$, from (4.24), we obtain that

$$\gamma_{k+1}^{-\frac{1-\varepsilon}{1+\varepsilon}} \geqslant \gamma_k^{-\frac{1-\varepsilon}{1+\varepsilon}} \left(1 - c\gamma_k^{\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{1-\varepsilon}{1+\varepsilon}}.$$

Noting that

$$(1 - x)^{-\frac{1-\varepsilon}{1+\varepsilon}} \geqslant 1 + \frac{1-\varepsilon}{1+\varepsilon}x \quad \forall x \in (0, 1),$$

we have that

$$\gamma_{k+1}^{-\frac{1-\varepsilon}{1+\varepsilon}} \geqslant \gamma_k^{-\frac{1-\varepsilon}{1+\varepsilon}} + c\frac{1-\varepsilon}{1+\varepsilon}.$$

It follows that

$$\gamma_k \leqslant \left(ck\frac{1-\varepsilon}{1+\varepsilon} + \gamma_0^{-\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{1+\varepsilon}{1-\varepsilon}} \leqslant \left(ck\frac{1-\varepsilon}{1+\varepsilon} + \rho^{-\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \quad k = 0, 1, \ldots.$$

□

For the critical index $\varepsilon = 0$, we obtain

Theorem 4.6. *Let Y be a general Banach space, and X be a Banach space which is p-convex and q-smooth with $1/p + 1/q = 1$. Assume there exists a solution $x^\dagger$ to (3.1) and that assumption 4.2 holds with $\varepsilon = 0$. Furthermore, assume that*

$$\|DF(x)\| \leqslant \hat{L} \quad \forall x \in \mathcal{B}, \tag{4.27}$$

and that the stability constant C_F and the positive stepsize μ satisfy the inequality

$$\mu^{q-1} < \frac{q}{G_q\hat{L}^q}\left(1 - \frac{1}{2}LC_F^2\left(\frac{p}{C_p}\right)^{\frac{2}{p}}\right). \tag{4.28}$$

Then the iterates satisfy

$$\Delta_p(x_k, x^\dagger) \to 0 \quad \textit{as} \quad k \to \infty.$$

Moreover, let

$$c = \mu\left(\frac{G_q}{q}\mu^{q-1}\hat{L}^q - 1 + \frac{1}{2}LC_F^2\left(\frac{p}{C_p}\right)^{\frac{2}{p}}\right)C_F^{-2p}. \tag{4.29}$$

The convergence rate is given by

$$\Delta_p(x_k, x^\dagger) \leqslant (\Delta_p(x_0, x^\dagger)^{-1} + ck)^{-1}. \tag{4.30}$$

Proof. Using (4.17) in the proof of theorem 4.5 subject to the substitution $\varepsilon = 0$, we obtain that

$$\gamma_{k+1} - \gamma_k \leqslant \left(\frac{G_q}{q}\mu^q \hat{L}^q - \mu + \frac{\mu}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\right)\|F(x_k) - y\|^p.$$

Note that, by (4.28), the right-hand side of the above inequality is non-positive and $0 < c < 1$. Then, using the Hölder-type stability (4.2) with $\varepsilon = 0$, we have that

$$\gamma_{k+1} \leqslant \gamma_k - c\gamma_k^2. \tag{4.31}$$

The convergence result and convergence rate (4.30) can be deduced by using the same arguments as in the proof of theorem 4.5. □

Remark 4.7. The Hölder-type stability condition (3.3) or (4.2) is implied by a lower bound of the Fréchet derivative DF. More precisely, if, there exists a constant C such that

$$\left\|DF(x)\left(\frac{x - x^\dagger}{\|x - x^\dagger\|}\right)\right\| \geqslant C\|x - x^\dagger\|^{1-\alpha} \quad \forall x \in B_r(x^\dagger) \cap \mathcal{D}(F),$$

for some $\alpha \in (0, 1]$ and r sufficiently small, then, by combining this and

$$\|F(\tilde{x}) - F(x) - DF(x)(\tilde{x} - x)\| \leqslant \frac{L}{2}\|\tilde{x} - x\|^2 \quad \forall x, \tilde{x} \in \mathcal{D}(F)$$

we obtain that

$$\|x - x^\dagger\| \leqslant C_F\|F(x) - F(x^\dagger)\|^{\frac{1}{2-\alpha}} \quad \forall x \in B_r(x^\dagger) \cap \mathcal{D}(F),$$

for some constant C_F depending on C and L. The ill-posedness of many inverse problems indicates that in general it is impossible to obtain a lower bound for DF. If one projects the forward operator F properly, an estimate for the lower bound of DF could be obtained. Under various conditions, the lower bound for DF has been investigated in the analysis of inverse problems. For example, see [13, 32, 17] for the EIT problem, [9] for the inverse medium problem associated with the Helmholtz equation and [7] for the inverse medium problem for electromagnetic waves.

5. Example: electrical impedance tomography

In this section, we discuss Calderón's inverse problem, which forms the mathematical foundation of the EIT problem [13]. For a recent review, we refer to [34]. We mention some key uniqueness results, namely [23, 24, 33, 8]. Here, we focus on results pertaining to stability; see [3–5]. In particular, we relate to the work of Alessandrini and Vesella [6] and Beretta and Francini [10], who establish a Lipschitz-type stability estimate if the conductivity is piecewise constant on a finite number of subdomains with jumps, for the real-valued and complex-valued cases, respectively.

5.1. The Dirichlet-to-Neumann map

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. Given a potential $f \in H^{1/2}(\partial\Omega)$ on the boundary, the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem:

$$\begin{cases} \nabla\cdot(\gamma\nabla u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet-to-Neumann map, or voltage-to-current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega},$$

where ν denotes the unit outer normal vector to $\partial\Omega$.

The forward operator F is defined by

$$\begin{aligned} F: \quad X \subset L^\infty_+(\Omega) &\to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)), \\ \gamma &\mapsto \Lambda_\gamma. \end{aligned} \tag{5.1}$$

The Fréchet derivative DF of F at $\gamma = \gamma_0$ is given by

$$\begin{aligned} DF(\gamma_0): \quad X \subset L^\infty(\Omega) &\to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \\ \delta\gamma &\mapsto DF(\gamma_0)(\delta\gamma), \end{aligned} \tag{5.2}$$

and $DF(\gamma_0)(\delta\gamma)$ is given by

$$\langle DF(\gamma_0)(\delta\gamma) f, g\rangle = \int_\Omega \delta\gamma \nabla u \cdot \nabla v \, \mathrm{d}x, \qquad f, g \in H^{1/2}(\partial\Omega), \tag{5.3}$$

where

$$\begin{cases} \nabla\cdot(\gamma_0\nabla u) = \nabla\cdot(\gamma_0\nabla v) = 0, & \text{in } \Omega, \\ u = f, \quad v = g & \text{on } \partial\Omega. \end{cases}$$

We note that $L^\infty(\Omega)$ is not a uniformly convex Banach space. Furthermore, to obtain the Hölder-type stability, the pre-image space needs to be reduced. We specify the proper space X in subsection 5.3.

For $n = 2$, Astala and Päivärinta proved that Λ_γ uniquely determines γ under the assumption that $\gamma \in L^\infty(\Omega)$. For $n \geqslant 3$, Päivärinta *et al* proved the uniqueness under the assumption that $\gamma \in W^{3/2,\infty}(\Omega)$ [28].

5.2. Lipschitz stability

It is possible to obtain Lipschitz-type stability, essentially, by assuming that γ belongs to a particular finite-dimensional space.

We write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ for $n \geqslant 2$. With $B_R(x)$, $B'_R(x')$ and $Q_R(x)$ we denote, respectively, the open ball in $\mathbb{R}^n$ centered at x of radius R, the ball in $\mathbb{R}^{n-1}$ centered at x' of radius R and the cylinder $B'_R(x') \times (x_n - R, x_n + R)$. For the simplicity of notation, $B_R(0)$, $B'_R(0)$ and $Q_R(0)$ are denoted by B_R, B'_R and Q_R, respectively.

Definition 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. We say that $\partial\Omega$ is of Lipschitz class with constants $r_0, L > 0$, if for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates such that $P = 0$ and*

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \phi(x')\},$$

where ϕ is a Lipschitz continuous function on B'_{r_0} with $\phi(0) = 0$ and

$$\|\phi\|_{C^{0,1}(B'_{r_0})} \leqslant L r_0.$$

Definition 5.2. *Let Ω be a bounded domain in $\mathbb{R}^n$. Given $\alpha \in (0, 1)$, we say that $\partial\Omega$ is of $C^{1,\alpha}$ class with constants $r_0, L > 0$, if for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates such that $P = 0$ and*

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \phi(x')\},$$

where ϕ is a $C^{1,\alpha}$ function on B'_{r_0} with $\phi(0) = |\nabla\phi(0)| = 0$ and

$$\|\phi\|_{C^{1,\alpha}(B'_{r_0})} \leqslant L r_0.$$

Assumption 5.3. *$\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying*

$$|\Omega| \leqslant A|B_{r_0}|.$$

Here and in the following $|\Omega|$ denotes the Lebesgue measure of Ω. We assume that $\partial\Omega$ is of Lipschitz class with constants r_0 and L.

Assumption 5.4. *The conductivity γ is a piecewise constant function of the form*

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x),$$

satisfying the ellipticity condition

$$K^{-1} \leqslant \gamma \leqslant K$$

for some constant K, where γ_j, $j = 1, \ldots, N$ are unknown real numbers and D_j are known open sets in $\mathbb{R}^n$.

Assumption 5.5. *The D_j, $j = 1, \ldots, N$ are connected and pairwise non-overlapping open sets such that $\cup_{j=1}^{N} \overline{D}_j = \overline{\Omega}$ and ∂D_j are of $C^{1,\alpha}$ class with constants r_0 and L for all $j = 1, \ldots, N$. We also assume that there exists one region, say D_1, such that $\partial D_1 \cap \partial\Omega$ contains an open portion Σ_1 of $C^{1,\alpha}$ class with constants r_0 and L. For every $j \in \{2, \ldots, N\}$ there exist $j_1, \ldots, j_M \in \{1, \ldots, N\}$ such that*

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \ldots, M$,

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a open portion Σ_k of $C^{1,\alpha}$ class with constants r_0 and L.

Alessandrini and Vessella [6] establish the following Lipschitz stability estimate.

Theorem 5.6. *Let Ω satisfy assumption 5.3 and $\gamma^{(k)}$, $k = 1, 2$ be two real piecewise constant functions satisfying assumption 5.4 and D_j, $j = 1, \ldots, N$ satisfying assumption 5.5. Then there exists a constant $C = C(n, r_0, L, A, K, N)$ such that*

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leqslant C\|\Lambda_{\gamma^{(1)}} - \Lambda_{\gamma^{(2)}}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}. \tag{5.4}$$

5.3. Convergence

We verify that the assumptions of section 4 can be satisfied. We specify our pre-image space as

$$X = \operatorname{span}\{\chi_{D_1}, \ldots, \chi_{D_N}\} \tag{5.5}$$

equipped with L^p-norm, $p > 1$. With the aid of this particular basis of X, one can show that F and DF are Lipschitz continuous. Moreover, assuming that γ_1 and γ_2 satisfy assumption 5.4 and Ω satisfies assumption 5.3, we have the following estimates:

$$\begin{aligned}
&\|F(\gamma_1) - F(\gamma_2)\|_{\mathcal{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))} \leqslant C\|\gamma_1 - \gamma_2\|_{L^p(\Omega)},\\
&\|DF\|_{\mathcal{L}(X, \mathcal{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega)))} \leqslant \hat{L},\\
&\|DF(\gamma_1) - DF(\gamma_2)\|_{\mathcal{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))} \leqslant L\|\gamma_1 - \gamma_2\|_{L^p(\Omega)},
\end{aligned} \tag{5.6}$$

where C, $\hat{L}$ and L depend on Ω, N and ellipticity constant K, respectively. Furthermore, since X is finite dimensional, the weak topology is equivalent to the strong topology. Hence, F is a weakly sequentially closed operator.

Let Ω satisfy assumption 5.3, pre-image space X be defined by (5.5) and F be defined by (5.1). Assume that $y = F(\gamma^\dagger)$ for some $\gamma^\dagger \in X$. Then assumptions 4.1 and (4.3) of theorem 4.5 are satisfied. Hence the Landweber iteration (2.12) converges with the convergence radius given by (4.5) and the convergence rate is given by (4.8). Convergence of a regularized Newton method for a finite-dimensional EIT problem was proven by Lechleitner and Rieder [26]. Their analysis, however, is based on the tangential cone condition.

6. Discussion

We discuss a Landweber iteration method for solving nonlinear operator equations in both Hilbert and Banach spaces. Traditionally, the gradient-type methods are often regarded as too slow for practical applications. Provided that the nonlinearity of the forward operator obeys a Hölder-type stability, we could prove the convergence and give a sublinear convergence rate. With a Lipschitz-type stability, the convergence rate switches to a linear one. Based on these convergence rates, we anticipate that this Landweber iteration is a valuable tool in solving inverse problems in both Hilbert and Banach spaces. This also motivates the study of Hölder/Lipschitz-type stability in inverse problems to provide explicit reconstructions.

Acknowledgments

The research in this paper was initiated at the Program on Inverse Problems and Applications at MSRI, Berkeley, during the Fall of 2010. This research was supported in part by National Science Foundation grant CMG DMS-1025318 and in part by the members of the Geo-Mathematical Imaging Group at Purdue University. The work of OS was supported by the Austrian Science Fund (FWF) within the national research network Photoacoustic Imaging in Biology and Medicine, project S10505-N20.

References

[1] Alber Y I and Butnariu D 1997 Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces *J. Optim. Theory Appl.* **92** 33–61

[2] Alber Y I, Kartsatos A G and Litsyn E 1996 Iterative solution of unstable variational inequalities on approximately given sets *Abstr. Appl. Anal.* **1** 45–64

[3] Alessandrini G 1988 Stable determination of conductivity by boundary measurements *Appl. Anal.* **27** 153–72

[4] Alessandrini G 1991 Determining conductivity by boundary measurements, the stability issue *Applied and Industrial Mathematics* (*Math. Appl.* vol 56) (Dordrecht: Kluwer Acad. Publ.) pp 317–24

[5] Alessandrini G 2007 Open issues of stability for the inverse conductivity problem *J. Inverse Ill-Posed Probl.* **15** 451–60

[6] Alessandrini G and Vessella S 2005 Lipschitz stability for the inverse conductivity problem *Adv. Appl. Math.* **35** 207–41

[7] Ammari H and Bao G 2001 Analysis of the scattering map of a linearized inverse medium problem for electromagnetic waves *Inverse Problems* **17** 219–34

[8] Astala K and Päivärinta L 2006 Calderón's inverse conductivity problem in the plane *Ann. Math.* **163** 265–99

[9] Bao G, Chen Y and Ma F 2000 Regularity and stability for the scattering map of a linearized inverse medium problem *J. Math. Anal. Appl.* **247** 255–71

[10] Beretta E and Francini E 2011 Lipschitz stability for the electrical impedance tomography problem: the complex case *Commun. Partial Differ. Eqns* **36** 1723–49

[11] Bregman L M 1967 The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Comput. Math. Math. Phys.* **7** 200–17

[12] Butnariu D, Iusem A N and Resmerita E 2000 Total convexity for powers of the norm in uniformly convex Banach spaces *J. Convex Anal.* **7** 319–34

[13] Calderón A-P 1980 On an inverse boundary value problem *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro) Soc. Brasil. Mat.* pp 65–73

[14] Chidume C E and Zegeye H 2001 Global iterative schemes for accretive operators *J. Math. Anal. Appl.* **257** 364–77

[15] Cioranescu I 1990 *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems* (*Mathematics and Its Applications* vol 62) (Dordrecht: Kluwer)

[16] Curtis F E, Schenk O and Wächter A 2010 An interior-point algorithm for large-scale nonlinear optimization with inexact step computations *SIAM J. Sci. Comput.* **32** 3447–75

[17] Dobson D C 1992 Estimates on resolution and stabilization for the linearized inverse conductivity problem *Inverse Problems* **8** 71–81

[18] Hanke M, Neubauer A and Scherzer O 1995 A convergence analysis of the Landweber iteration for nonlinear ill-posed problems *Numer. Math.* **72** 21–37

[19] Hein T and Kazimierski K S 2010 Modified Landweber iteration in Banach spaces—convergence and convergence rates *Numer. Funct. Anal. Optim.* **31** 1158–84

[20] Hofmann B, Kaltenbacher B, Pöschl C and Scherzer O 2007 A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators *Inverse Problems* **23** 987

[21] Kaltenbacher B, Neubauer A and Scherzer O 2008 *Iterative Regularization Methods for Nonlinear Ill-posed Problems* (*Radon Series on Computational and Applied Mathematics* vol 6) (Berlin: Walter)

[22] Kaltenbacher B, Schöpfer F and Schuster T 2009 Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems *Inverse Problems* **25** 065003

[23] Kohn R and Vogelius M 1984 Determining conductivity by boundary measurements *Commun. Pure Appl. Math.* **37** 289–98

[24] Kohn R and Vogelius M 1985 Determining conductivity by boundary measurements: II. Interior results *Commun. Pure Appl. Math.* **38** 643–67

[25] Landweber L H 1951 An iteration formula for Fredholm integral equations of the first kind *Am. J. Math.* **73** 615–24

[26] Lechleiter A and Rieder A 2008 Newton regularizations for impedance tomography: convergence by local injectivity *Inverse Problems* **24** 065009

[27] Osher S, Burger M, Goldfarb D, Xu J and Yin W 2005 An iterative regularization method for total variation-based image restoration *Multiscale Model. Simul.* **4** 460–89

[28] Päivärinta L, Panchenko A and Uhlmann G 2003 Complex geometrical optics solutions for Lipschitz conductivities *Rev. Mat. Iberoamericana* **19** 57–72

[29] Scherzer O, Grasmair M, Grossauer H, Haltmeier M and Lenzen F 2009 *Variational Methods in Imaging* (*Applied Mathematical Sciences* vol 167) (New York: Springer)

[30] Schöpfer F, Louis A K and Schuster T 2006 Nonlinear iterative methods for linear ill-posed problems in Banach spaces *Inverse Problems* **22** 311

[31] Schöpfer F, Schuster T and Louis A K 2008 Metric and Bregman projections onto affine subspaces and their computation via sequential subspace optimization methods *J. Inverse Ill-Posed Probl.* **16** 479–506

[32] Somersalo E, Isaacson D and Cheney M 1992 A linearized inverse boundary value problem for Maxwell's equations *J. Comput. Appl. Math.* **42** 123–36

[33] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem *Ann. Math.* **125** 153–69

[34] Uhlmann G 2009 Electrical impedance tomography and Calderón's problem *Inverse Problems* **25** 123011

[35] Xu Z B and Roach G F 1991 Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces *J. Math. Anal. Appl.* **157** 189–210

[36] Zeidler E 1990 *Nonlinear Functional Analysis and its Applications II/A. Linear Monotone Operators* (New York: Springer)

[37] Zeidler E 1990 *Nonlinear Functional Analysis and its Applications II/B. Nonlinear Monotone Operators* (New York: Springer)