UNIFORM ASYMPTOTIC EXPANSION OF THE SQUARE-ROOT HELMHOLTZ OPERATOR AND THE ONE-WAY WAVE PROPAGATOR*

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Abstract. The Bremmer coupling series solution of the wave equation, in generally inhomogeneous media, requires the introduction of pseudodifferential operators. Such operators appear in the diagonalization process of the acoustic system's matrix of partial differential operators upon extracting a principal direction of (one-way) propagation. In this paper, in three dimensions, uniform asymptotic expansions of the Schwartz kernels of these operators are derived. Also, we derive a uniform asymptotic expansion of the one-way propagator appearing in the series. We focus on designing closed-form representations, valid in the high-frequency limit, taking into account *critical scattering-angle* phenomena. The latter phenomena are not dealt with in the standard calculus of pseudodifferential operators. Our expansion is not limited by propagation angle. In principle, the uniform asymptotic expansion of a kernel follows by matching its asymptotic behaviors away and near its diagonal.

1. Introduction. Directional wave field decomposition is a tool for analyzing and computing wave propagation in configurations with a special directionality, such as a waveguiding structure. Such a method consists of three main steps: (i) decomposing the field into two constituents, propagating "one-way" upward or downward along a preferred or principal direction, (ii) computing the interaction of the counterpropagating constituents, and (iii) recomposing the constituents into observables at the positions of interest. The Bremmer series [1] then synthesizes the constituents into a full-wave solution. Each term in the series represents a wave constituent that has traveled up and down along the principal direction a number of times equal to its order. Thus we are able to trace waves: evolution is no longer in time but now in the vertical coordinate, vertical being identified with the principal direction. The microlocal analysis of the one-way wave propagator can be found in Treves [2].

Applications of the generalized Bremmer series solution to the wave equation include (i) the identification and elimination of multiple scattered wave constituents and (ii) the formulation of various imaging and inverse scattering procedures in remote sensing. In general, the inverse scattering problem can be decomposed into a coupled inverse "contrast-source" or "reflectivity"–inverse "constituency" problem. With the aid of time-reversal mirrors, each pair of successive terms in the Bremmer series can be exploited to construct the reflectivity (see de Hoop [3]).

The generalized Bremmer series can be viewed as a full-wave extension of the (high-frequency) geometrical ray series representation of the wave field embedded

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in the Kirchhoff approximation (see Frazer [4]). The Maslov canonical operator is replaced by a Trotter product (see de Hoop, Le Rousseau, and Biondi [5]). The Bremmer series–Trotter product approach encompasses the microlocal, Kirchhoff–Maslov, representation of the wave field. Extensive lists of references to applications of the generalized Bremmer series in exploration and crustal seismology, ocean acoustics, and integrated optics can be found in Van Stralen, de Hoop, and Blok [6].

De Hoop [1] originally formulated the generalized Bremmer series modeling method in the time-*Laplace* domain. Owing to the fact that the medium can vary in the directions transverse to the preferred direction, pseudodifferential calculus became a natural tool to introduce the up and downgoing Green's functions: pseudodifferential operators appear in the directional (de)composition, in the downward and upward propagation or continuation, and in the interaction (reflection and transmission) between the counterpropagating constituents due to variations in medium properties in the preferred direction. The time-Laplace domain is not amenable to computations, however.

Various approaches have been developed over the years in the time-*Fourier* domain to approximate the operators appearing in the Bremmer series to make numerical computations feasible. An overview of the approaches based on rational (paraxial) expansions of the operator symbols can be found in Van Stralen, de Hoop, and Blok [6]. An overview of approaches based on phase-screen-like approximations of the operator symbols can be found in de Hoop, Le Rousseau, and Wu [7]. With these numerical approaches, however, critical "scattering-angle" phenomena such as the ones associated with rays the tangents to which become horizontal (for example, turning rays) cannot be modeled. With the approach proposed in this paper, this limitation is removed. In particular media, spectral analysis can be employed to find exact time-Fourier representations of mentioned operators (see Fishman, de Hoop, and van Stralen [8]).

In this paper, our goal is to gain analytic insight into the propagation and scattering of waves as described by the generalized Bremmer series—while developing a time-*Fourier* analysis of the constituent operators. We extend earlier results (Fishman, Gautesen, and Sun [9] and de Hoop and Gautesen [10]) in this direction that were derived in two dimensions to three (and higher) dimensions. Instead of using pseudodifferential operators in the time-Laplace domain, we will here employ microlocal and uniform asymptotics techniques combined in the time-Fourier domain. We focus our analysis on the development of a uniform asymptotic expansion of the transversal part of the one-way wave operator kernel (of the square-root Helmholtz type) and the associated one-way wave propagator. For the completion of the Bremmer coupling series we refer the reader to our earlier paper.

The uniform asymptotic expansions also provide the basis for a numerical scheme. Such a scheme would involve the computations of (i) a spatially varying effective index of refraction and (ii) a spatially varying effective "distance" in the transverse directions, and then applying the kernel. The effective index of refraction and the effective metric are computed along the bicharacteristics constrained to the plane spanned by transverse directions.

The outline of this paper is as follows. In the next section a summary of the method of directional decomposition, leading to a coupled system of one-way wave equations is given. In section 3, the medium is decomposed into thin slabs. In each thin slab we introduce a "characteristic" Green's function. In section 4 we introduce representations of the square-root operator and the one-way wave propagator in terms of the characteristic Green's function. The key effort is developing a uniform

asymptotic expansion of the characteristic Green's function. Such an expansion in the absence of transverse caustics is developed in section 5 and in the presence of transverse caustics in section 6. In both cases an "inner" (near-field) and "outer" (far-field) representation is derived upon which a matching procedure in a boundary layer is invoked. The latter synthesizes the uniformly valid expression. Section 7 summarizes the main result of the paper: the uniform asymptotic expansion for the square-root operator and the likewise expansion for the one-way wave propagator in higher dimensions. We conclude with a discussion (section 8).

2. Directional wave field decomposition. For the details on the derivation of the Bremmer coupling series solution of the acoustic wave equation, we refer the reader to de Hoop [1]. Here, we restrict ourselves to a summary of this wave field decomposition method.

Notation, transformations. We consider acoustic waves in a three-dimensional configuration. In this configuration, let p denote the pressure and $(v_{1,2}, v_3) = (v_1, v_2, v_3)$ the particle velocity. We introduce the Fourier transformation with respect to time t as

$$(2.1) \quad (\mathcal{F}\{p, v_{1,2}, v_3\})(x_{1,2}, x_3, \omega) = \int_{t \in \mathbb{R}_{\geq 0}} \{p, v_{1,2}, v_3\}(x_{1,2}, x_3, t) \ \exp(\mathrm{i}\omega t)\, \mathrm{d}t$$

for $\mathrm{Im}\{\omega\} > 0$. Under this transformation, assuming zero initial conditions, we have $\partial_t \to -\mathrm{i}\omega$.

In each subdomain of the configuration where the acoustic properties vary continuously with position, the acoustic wave field $\{p, v_{1,2}, v_3\}$ satisfies the system of partial differential equations

$$(2.2) \qquad\qquad\qquad \partial_k p - \mathrm{i}\omega\rho\, v_k = f_k,$$

$$(2.3) \qquad\qquad -\mathrm{i}\omega\kappa\, p + \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = q.$$

Here, ρ denotes the volume density of mass, κ the compressibility, q the volume source density of injection rate, and f_k the volume source density of force.

The spatial variation of the wave field along a direction of preference can now be expressed in terms of the variation of the wave field in the direction perpendicular to it. The direction of preference or principal direction is taken (globally) along the x_3-axis (or "vertical" axis) and the remaining ("transverse" or "horizontal") coordinates are denoted by (x_1, x_2) or $x_{1,2}$.

The reduced system of equations. Directional decomposition requires a separate handling of the horizontal or transverse component of the particle velocity. From (2.2) and (2.3) we obtain

$$(2.4) \qquad\qquad v_{1,2} = -\mathrm{i}\rho^{-1}\omega^{-1}(\partial_{1,2}p - f_1),$$

leaving, upon substitution, the matrix differential equation $(I, J = 1, 2)$

$$(2.5) \quad (\partial_3 \delta_{IJ} - \mathrm{i}\omega A_{IJ})\, F_J = N_I, \quad A_{IJ} = A_{IJ}(x_{1,2}, D_{1,2}; x_3), \quad D_{1,2} \equiv -\frac{\mathrm{i}}{\omega}\partial_{1,2},$$

in which the elements of the acoustic field matrix[1] are given by

$$(2.6) \qquad\qquad\qquad F_1 = p, \qquad F_2 = v_3,$$

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[1] Present ocean-bottom seismic acquisition technology allows both p and v_3 to be measured.

the elements of the acoustic system's matrix operator by

$$(2.7) \qquad A_{11} = A_{22} = 0,$$

$$(2.8) \qquad A_{12} = \rho,$$

$$(2.9) \qquad A_{21} = -D_1(\rho^{-1}D_1) - D_2(\rho^{-1}D_2) + \kappa,$$

and the elements of the notional source matrix by

$$(2.10) \qquad N_1 = f_3, \qquad N_2 = D_1(\rho^{-1}f_1) + D_2(\rho^{-1}f_2) + q.$$

It is observed that the right-hand side of (2.4) and A_{IJ} contain the spatial derivatives $D_{1,2}$ with respect to the horizontal coordinates only. In the sequel of the paper it will become clear that $D_{1,2}$ has the interpretation of *horizontal slowness* operator. Further, it is noted that A_{12} is simply a multiplicative operator.

The coupled system of one-way wave equations. To distinguish up and downgoing constituents in the wave field, we shall construct an appropriate linear operator L_{IJ} with

$$(2.11) \qquad F_I = L_{IJ}W_J,$$

which, with the aid of the commutation relation ($[.,.]$ denotes the commutator)

$$(2.12) \qquad (\partial_3 L_{IJ}) = [\partial_3, L_{IJ}],$$

transforms (2.5) into

$$(2.13) \qquad L_{IJ}(\partial_3 \delta_{JM} - \mathrm{i}\omega\Lambda_{JM})W_M = -(\partial_3 L_{IJ})W_J + N_I.$$

Transformation (2.11) should result in the diagonalization of the operator A_{IJ} in the sense that

$$(2.14) \qquad A_{IJ}L_{JM} = L_{IJ}\Lambda_{JM},$$

where Λ_{JM} is a diagonal matrix of operators. We denote L_{IJ} as the composition operator and W_M as the wave column matrix. The expression in parentheses on the left-hand side of (2.13) represents the two so-called *one-way* wave operators. The first term on the right-hand side of (2.13) is representative for the scattering due to variations of the medium properties in the vertical direction. The diffraction due to variations of the medium properties in the horizontal directions is contained in Λ_{JM} and, implicitly, in L_{IJ}. This diffraction comprises the multipathing of characteristics that commonly occurs in geophysical configurations.

To investigate whether solutions of (2.14) exist, we introduce the column matrix operators $L_I^{(\pm)}$ according to

$$(2.15) \qquad L_I^{(+)} = L_{I1}, \qquad L_I^{(-)} = L_{I2}.$$

Upon writing the diagonal elements of Λ_{JM} as

$$(2.16) \qquad \Lambda_{11} = \Gamma^{(+)}, \qquad \Lambda_{22} = \Gamma^{(-)},$$

(2.14) decomposes into the two systems of equations

$$(2.17) \qquad A_{IJ}L_J^{(\pm)} = L_I^{(\pm)}\Gamma^{(\pm)}.$$

By analogy with the case where the medium is translationally invariant in the horizontal directions, we shall denote $\Gamma^{(\pm)}$ as the *vertical slowness* operators. Notice that the operators $L_1^{(\pm)}$ synthesize the acoustic pressure and that the operators $L_2^{(\pm)}$ synthesize the vertical particle velocity. Through mutual elimination, the equations for $L_1^{(\pm)}$ and $L_2^{(\pm)}$ can be decoupled as follows:

$$(2.18) \qquad A_{12}A_{21}L_1^{(\pm)} = L_1^{(\pm)}\Gamma^{(\pm)}\,\Gamma^{(\pm)},$$

$$(2.19) \qquad A_{21}A_{12}L_2^{(\pm)} = L_2^{(\pm)}\Gamma^{(\pm)}\,\Gamma^{(\pm)}.$$

The partial differential operators on the left-hand sides differ from one another in the case where the volume density of mass does vary in the horizontal directions.

To ensure that nontrivial solutions of (2.18)–(2.19) exist, one equation must imply the other. To construct a formal solution, an ansatz is introduced in the form of a commutation relation for one of the components $L_J^{(\pm)}$ that restricts the freedom in the choice for the other component. In the *acoustic-pressure normalization* analogue one assumes that $L_2^{(\pm)}$ can be chosen such that

$$(2.20) \qquad [A_{12}L_2^{(\pm)}, A_{12}A_{21}] = 0.$$

In view of (2.19), $\Gamma^{(\pm)}$ must then satisfy

$$(2.21) \qquad A_{12}A_{21} - \Gamma^{(\pm)}\,\Gamma^{(\pm)} = 0.$$

The commutation relation for $L_1^{(\pm)}$ follows as $[L_1^{(\pm)}, A_{12}A_{21}] = 0$ and a possible solution of (2.17) is

$$(2.22) \qquad L_2^{(\pm)} = A_{12}^{-1}\Gamma^{(\pm)}, \qquad L_1^{(\pm)} = I.$$

Since $L_2^{(\pm)}$ as given by (2.22) satisfies (2.20), the ansatz is justified. The solutions of (2.21) are written as

$$(2.23) \qquad \Gamma^{(+)} = -\Gamma^{(-)} = \Gamma = A^{1/2} \quad \text{with} \ \ A = A_{12}A_{21}.$$

Thus, the *composition* operator becomes

$$(2.24) \qquad L = \begin{pmatrix} I & I \\ A_{12}^{-1}\Gamma & -A_{12}^{-1}\Gamma \end{pmatrix}.$$

Note that we have decomposed the pressure field according to

$$F_1 = F_1^{(+)} + F_1^{(-)} \quad \text{with} \ \ F_1^{(+)} = W_1, \quad F_1^{(-)} = W_2.$$

In terms of the inverse vertical slowness operator, $\Gamma^{-1} = A^{-1/2}$, the *decomposition* operator then follows as

$$(2.25) \qquad L^{-1} = \tfrac{1}{2}\begin{pmatrix} I & \Gamma^{-1}A_{12} \\ I & -\Gamma^{-1}A_{12} \end{pmatrix}.$$

Using the decomposition operator, (2.13) transforms into

$$(2.26) \qquad (\partial_3\delta_{IM} - i\omega\Lambda_{IM})\,W_M = -(L^{-1})_{IM}(\partial_3 L_{MJ})W_J + (L^{-1})_{IM}N_M,$$

which can be interpreted as a coupled system of one-way wave equations. The propagation is captured by the left-hand side. The coupling between the counter-propagating components, W_1 and W_2, is apparent in the first source-like term on the right-hand side. The waves are excited by the second term on the right-hand side. We have

$$(2.27) \qquad -L^{-1}(\partial_3 L) = \begin{pmatrix} T & R \\ R & T \end{pmatrix},$$

in which T and R represent the *transmission* and *reflection* operators, respectively: let $Y = A_{12}^{-1}\Gamma$ denote the *admittance* operator; then

$$(2.28) \qquad R = -T = \tfrac{1}{2}Y^{-1}(\partial_3 Y).$$

The two-way Helmholtz equation. Suppose that the medium does *not* vary with x_3. Eliminating F_2 or v_3 from (2.5) then leads to the second-order equation for the pressure,

$$(2.29) \qquad [\partial_3^2 + \omega^2 A(x_{1,2}, D_{1,2})] F_1 = \mathrm{i}\omega\, \rho N_2 + \partial_3 N_1,$$

the *two-way* Helmholtz equation, where A is given by (2.23).

3. Decomposition of the configuration into thin slabs. We will now decompose the *medium* into (thin) slabs. Each slab in our three-dimensional configuration is assumed to be *invariant* in the direction of preference, x_3: the compressibility, κ, may vary in the transverse directions, whereas the density is assumed to be constant all together. However, the medium may vary from slab to slab, and hence the vertical coordinate x_3 becomes a parameter that identifies the slab in our further analysis.

The characteristic operator. As mentioned, in our thin-slab analysis, we will consider the following medium profile:

$$(3.1) \qquad \rho = \text{const.},$$
$$(3.2) \qquad \kappa(x_{1,2}) = \kappa_0\, n^2(x_{1,2});$$

thus, setting $\kappa_0 = \rho^{-1}c_0^{-2}$, the wave speed follows from

$$c^{-2}(x_{1,2}) = c_0^{-2}n^2(x_{1,2}),$$

where n denotes the index of refraction. The operator in (2.23) is then given by

$$(3.3) \qquad A(x_{1,2}, D_{1,2}) = -D_1^2 - D_2^2 + c_0^{-2}n^2(x_{1,2}).$$

We will denote A as the transverse Helmholtz or *characteristic* operator.

Factorization, Green's functions. We introduce the well-known Helmholtz equation and "characteristic" Green's function as (cf. (2.29))

$$(3.4) \qquad \begin{aligned} &[\partial_3^2 + \omega^2 A(x_{1,2}, D_{1,2})]\, G(x_{1,2}, x_3 - x_3'; x_{1,2}') \\ &= -\delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3'). \end{aligned}$$

The vertical slowness operators $\Gamma^{(\pm)}$ factorize the Helmholtz operator (cf. (2.23)):

$$(3.5) \quad \partial_3^2 + \omega^2 A(x_{1,2}, D_{1,2}) = [\partial_3 - \mathrm{i}\omega\, \Gamma^{(+)}(x_{1,2}, D_{1,2})]\,[\partial_3 - \mathrm{i}\omega\, \Gamma^{(-)}(x_{1,2}, D_{1,2})].$$

The one-way Green's functions $\mathcal{G}^{(\pm)}$ associated with the two factors satisfy

$$(3.6) \qquad \begin{aligned} &[\partial_3 - \mathrm{i}\omega\, \Gamma^{(\pm)}(x_{1,2}, D_{1,2})]\, \mathcal{G}^{(\pm)}(x_{1,2}, x_3 - x_3'; x_{1,2}') \\ &= \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3'). \end{aligned}$$

Vertical slowness as phase variable. Note that the Fourier representation of the causal Green's function G yields

$$(3.7) \quad G(x_{1,2}, x_3 - x_3'; x_{1,2}') = \frac{\omega}{2\pi c_0} \int_{\zeta \in \mathcal{Z}} \tilde{G}(x_{1,2}, x_{1,2}'; \zeta) \exp[\mathrm{i} \underbrace{(\omega/c_0)}_{k_0} |x_3 - x_3'| \zeta] \, \mathrm{d}\zeta.$$

Here, $\mathcal{Z}$ follows the real axis in the complex ζ-plane, below it for negative real parts and above it for positive real parts. Since

$$(3.8) \qquad \omega^2 \mathrm{A}(x_{1,2}, D_{1,2}) = \partial_1^2 + \partial_2^2 + (\omega/c_0)^2 \, n^2(x_{1,2}),$$

$\tilde{G}$ satisfies (cf. (3.4))

$$(3.9) \quad [\partial_1^2 + \partial_2^2 + (\omega/c_0)^2 \, (n^2(x_{1,2}) - \zeta^2)] \, \tilde{G}(x_{1,2}, x_{1,2}'; \zeta) = -\delta(x_1 - x_1')\delta(x_2 - x_2'),$$

or, more formally,

$$(3.10) \qquad -\omega^2 [\mathrm{A}(x_{1,2}, D_{1,2}) - c_0^{-2}\zeta^2] \, \tilde{G}(x_{1,2}, x_{1,2}'; \zeta) = \delta(x_1 - x_1')\delta(x_2 - x_2').$$

We can deform contour $\mathcal{Z}$ to a contour $\mathcal{Z}'$, say, such that the distance from a zero crossing of $n^2(x_{1,2}) - \zeta^2$ remains finite.

Observe the symmetry $\tilde{G}(x_{1,2}, x_{1,2}'; -\zeta) = \tilde{G}(x_{1,2}, x_{1,2}'; \zeta)$. Hidden inside the integral is a cut-off function in accordance with the microlocal representation of G.

4. Kernel representations in terms of the characteristic Green's function.

The one-way propagator. Using the image principle, we can express the one-way Green's functions in terms of the Green's function of the second-order Helmholtz equation,

$$(4.1)$$
$$\mathcal{G}^{(+)}(x_{1,2}, x_3 - x_3'; x_{1,2}') + \mathcal{G}^{(-)}(x_{1,2}, x_3 - x_3'; x_{1,2}') = -2\,\partial_3 G(x_{1,2}, x_3 - x_3'; x_{1,2}').$$

Hence, for $x_3 > x_3'$,

$$(4.2) \qquad \mathcal{G}^{(+)}(x_{1,2}, x_3 - x_3'; x_{1,2}') = -2\,\partial_3 G(x_{1,2}, x_3 - x_3'; x_{1,2}').$$

In fact, $\mathcal{G} \equiv \mathcal{G}^{(+)}$ is the kernel of the (upward) one-way wave propagator. In view of (4.2) this kernel satisfies the property

$$(4.3) \qquad \partial_3^{2j} \mathcal{G} = [-\omega^2 \mathrm{A}(x_{1,2}, D_{1,2})]^j \mathcal{G}, \qquad j = 1, 2, \dots,$$

for $x_3 > x_3'$. We will pay special attention to the so-called thin-slab expansion of $\mathcal{G}$.

The vertical slowness or square-root operator. The vertical slowness or square-root operator Γ (see (2.23)) acts on the wave field as

$$(4.4) \qquad (\Gamma\{W_1, W_2\})(x_{1,2}) = \int_{x_{1,2}' \in \mathbb{R}^2} \mathcal{C}(x_{1,2}, x_{1,2}') \, \{W_1, W_2\}(x_{1,2}') \, \mathrm{d}x_1' \mathrm{d}x_2',$$

where $\mathcal{C}$ denotes a Schwartz kernel. From this operator representation, we extract the left vertical slowness symbol through the Fourier transformation

$$(4.5) \qquad \gamma(x_{1,2}, p_{1,2}) = \int_{x_{1,2}' \in \mathbb{R}^2} \mathcal{C}(x_{1,2}, x_{1,2}') \exp[-\mathrm{i}\omega(x_\sigma - x_\sigma')p_\sigma] \, \mathrm{d}x_1' \mathrm{d}x_2',$$

where the summation convention has been invoked for $\sigma \in \{1, 2\}$. The left symbols of the horizontal slowness operators $D_{1,2}$ appear to be simply $p_{1,2}$. The relation between the left vertical slowness symbol and the horizontal slowness symbol constitutes the generalized slowness surface.

We will now focus on finding integral representations for the Schwartz kernel. First, note that the Schwartz kernel can be expressed in terms of the one-way Green's function,

$$(4.6) \qquad \mathcal{C}^{(+)}(x_{1,2}, x'_{1,2}; x'_3) = -\lim_{x_3 \downarrow x'_3} \frac{\mathrm{i}}{\omega} \partial_3 \mathcal{G}^{(+)}(x_{1,2}, x_3 - x'_3; x'_{1,2}),$$

$$(4.7) \qquad \mathcal{C}^{(-)}(x_{1,2}, x'_{1,2}; x'_3) = -\lim_{x_3 \uparrow x'_3} \frac{\mathrm{i}}{\omega} \partial_3 \mathcal{G}^{(-)}(x_{1,2}, x_3 - x'_3; x'_{1,2}).$$

With (4.2) we find that

$$(4.8) \qquad \mathcal{C}(x_{1,2}, x'_{1,2}; x'_3) = -\lim_{x_3 \downarrow x'_3} \frac{2}{\mathrm{i}\omega} \partial_3^2 G(x_{1,2}, x_3 - x'_3; x'_{1,2}).$$

Note that $\mathcal{C}$ is dependent on x'_3 through the index of refraction. We will suppress this dependence in our notation.

The inverse vertical slowness operator. The inverse or reciprocal vertical slowness operator admits the kernel identification

$$(4.9) \qquad \mathcal{A}_{-1/2}(x_{1,2}, x'_{1,2}) = -2\mathrm{i}\omega\, G(x_{1,2}, 0; x'_{1,2}).$$

From the inverse vertical slowness operator, the higher fractional powers of the characteristic operator can be obtained, viz., through the composition

$$(4.10) \qquad \mathcal{A}^{j-1/2} = \mathrm{A}^j \mathcal{A}_{-1/2}.$$

5. Uniform asymptotic expansion of the characteristic Green's function: The absence of caustics.

The inner solution. The inner region is determined by the condition

$$\|(x_1 - x'_1, x_2 - x'_2)\| = \mathcal{O}(k_0^{-1})$$

and corresponds to the behavior of the kernels near their diagonals. The inner region is so close to the "source" at $x'_{1,2}$ that caustics have not (yet) formed.

We reconsider (3.4),

$$[\partial_k \partial_k + k_0^2 n^2(x_{1,2})]\, G(x_{1,2}, x_3 - x'_3; x'_{1,2}) = -\delta(x_1 - x'_1)\delta(x_2 - x'_2)\delta(x_3 - x'_3)$$

and introduce the relative coordinate

$$y_j = x_j - x'_j, \qquad j \in \{1, 2, 3\}.$$

We expand the index of refraction about (x'_1, x'_2) according to

$$n^2(y_{1,2} + x'_{1,2}) = n^2(x'_{1,2}) + 2n(x'_{1,2})\,([y_1\partial_1 + y_2\partial_2]n)(x'_{1,2})$$
$$(5.1) \qquad + n(x'_{1,2})\,[y_1\partial_1 + y_2\partial_2]^2 n(x'_{1,2}) + (([y_1\partial_1 + y_2\partial_2]n)(x'_{1,2}))^2 + \dots,$$

where we differentiate $(\partial_{1,2})$ with respect to $x'_{1,2}$ while the argument of n and its derivatives is $x'_{1,2}$. We invoke the expansion of the Green's function in terms of $y_{1,2}$,

$$(5.2) \qquad G = G_0 + G_1 + G_2 + \dots,$$

where the subscript indicates the order in $y = (y_1^2 + y_2^2 + y_3^2)^{1/2}$. Then

$$
\begin{aligned}
&[\partial_k \partial_k + k_0^2 n^2(x'_{1,2})]\, G_0(x_{1,2}, x_3 - x'_3; x'_{1,2}) \\
(5.3) \qquad &\qquad = -\delta(x_1 - x'_1)\delta(x_2 - x'_2)\delta(x_3 - x'_3), \\
&[\partial_k \partial_k + k_0^2 n^2(x'_{1,2})]\, G_1(x_{1,2}, x_3 - x'_3; x'_{1,2}) \\
(5.4) \qquad &\qquad = -2k_0^2 n(x'_{1,2})\, ([y_1\partial_1 + y_2\partial_2]n)(x'_{1,2})\, G_0(x_{1,2}, x_3 - x'_3; x'_{1,2}), \\
&[\partial_k \partial_k + k_0^2 n^2(x'_{1,2})]\, G_2(x_{1,2}, x_3 - x'_3; x'_{1,2}) \\
(5.5) \qquad &\qquad = -2k_0^2 n(x'_{1,2})\, ([y_1\partial_1 + y_2\partial_2]n)(x'_{1,2})\, G_1(x_{1,2}, x_3 - x'_3; x'_{1,2}) \\
&\qquad\quad -k_0^2 \left[n(x'_{1,2})\, [y_1\partial_1 + y_2\partial_2]^2 n(x'_{1,2}) \right. \\
&\qquad\quad \left. +(([y_1\partial_1 + y_2\partial_2]n)(x'_{1,2}))^2 \right]\, G_0(x_{1,2}, x_3 - x'_3; x'_{1,2}),
\end{aligned}
$$

etc., with solutions obtained recursively as

$$(5.6)\, G_0(x_{1,2}, x_3 - x'_3; x'_{1,2}) = \frac{\exp[ik_0 n\, y]}{4\pi y},$$

$$(5.7)\, G_1(x_{1,2}, x_3 - x'_3; x'_{1,2}) = \tfrac{1}{2}ik_0 y\, ([y_1\partial_1 + y_2\partial_2]n)\, G_0(x_{1,2}, x_3 - x'_3; x'_{1,2}),$$

$$
\begin{aligned}
G_2(x_{1,2}, x_3 - x'_3; x'_{1,2}) = \frac{ik_0 y}{24n} \Big\{ & 4n[y_1\partial_1 + y_2\partial_2]^2 n \\
& + (3ik_0 ny + 1)([y_1\partial_1 + y_2\partial_2]n)^2 - y^2[(\partial_1 n)^2 + (\partial_2 n)^2] \\
(5.8)\qquad & + \frac{(\partial_1 n)^2 + (\partial_2 n)^2 - 2n[(\partial_1)^2 + (\partial_2)^2]n}{(ik_0 n)^2}(ik_0 ny - 1) \Big\} G_0.
\end{aligned}
$$

Inner expansion in midpoint coordinates. In the spirit of the Weyl calculus of kernel symbols (see [11, 21.6.5]), we can improve the above result by introducing the midpoint coordinates

$$\bar{x}_j = \tfrac{1}{2}(x_j + x'_j), \qquad j \in \{1, 2, 3\},$$

and re-expand the exponential according to

$$
\begin{aligned}
\exp[ik_0 n(x'_{1,2})\, y] = \exp[ik_0 n(\bar{x}_{1,2})\, y]\, \big\{ & 1 - \tfrac{1}{2}ik_0 y[y_1\partial_1 + y_2\partial_2]n \\
(5.9) \qquad & + \tfrac{1}{8}ik_0 y[y_1\partial_1 + y_2\partial_2]^2 n - \tfrac{1}{8}k_0^2 y^2([y_1\partial_1 + y_2\partial_2]n)^2 + \dots \big\},
\end{aligned}
$$

where the argument of n is now $\bar{x}_{1,2}$. The expansion for G (cf. (5.2) and (5.6)–(5.8)) can then be rewritten as

$$
\begin{aligned}
G = \frac{\exp[ik_0 n\, y]}{4\pi y} \Big\{ & 1 + \frac{ik_0 y}{24n} \Big[n[y_1\partial_1 + y_2\partial_2]^2 n + ([y_1\partial_1 + y_2\partial_2]n)^2 \\
& - y^2[(\partial_1 n)^2 + (\partial_2 n)^2] \\
(5.10) \qquad & + \frac{(\partial_1 n)^2 + (\partial_2 n)^2 - 2n[(\partial_1)^2 + (\partial_2)^2]n}{(ik_0 n)^2}(ik_0 ny - 1) \Big] + \dots \Big\}.
\end{aligned}
$$

Note that the odd order terms (up to this order, G_1) have disappeared. The expansion above has an improved error estimate, here $\mathcal{O}(y^4)$: We now have (cf. (5.2)) $G \simeq G_0 + G_2$ and upon substitution it follows that

$$(5.11) \qquad [\partial_k \partial_k + k_0^2 n^2(x_{1,2})] \, G = G_0(x_{1,2}, x_3 - x_3'; x_{1,2}') \, k_0^2 \mathcal{O}(y^4).$$

The outer solution. The outer region is determined by the condition

$$\|(x_1 - x_1', x_2 - x_2')\| = \mathcal{O}(1)$$

and corresponds to the behavior of the kernels away from their diagonals.

We reconsider (3.9),

$$[\partial_1^2 + \partial_2^2 + k_0^2(n^2(x_{1,2}) - \zeta^2)] \, \tilde{G}(x_{1,2}, x_{1,2}'; \zeta) = -\delta(x_1 - x_1')\delta(x_2 - x_2')$$

and we introduce the representation

$$(5.12) \qquad \tilde{G}(x_{1,2}, x_{1,2}'; \zeta) = \mathsf{C} \, \exp(\mathrm{i}k_0 \psi),$$

where C is a yet-to-be-determined constant. We expand ψ into phase and amplitude contributions,

$$(5.13) \qquad \psi = \phi + \underbrace{\frac{1}{\mathrm{i}k_0}\phi_1 + \frac{1}{(\mathrm{i}k_0)^2}\phi_2 + \dots}_{\text{amplitude}}.$$

Substituting this expansion into the partial differential equation, and collecting equal powers of $(\mathrm{i}k_0)$, results in the eikonal equation

$$(5.14) \qquad p^2 + q^2 + \zeta^2 - n^2(x_{1,2}) = 0,$$

for the leading order; here $p \equiv \partial_{x_1}\phi$ and $q \equiv \partial_{x_2}\phi$. The next order terms yield the equation

$$(5.15) \qquad 2p\,(\partial_{x_1}\phi_1) + 2q\,(\partial_{x_2}\phi_1) + \partial_{x_1}p + \partial_{x_2}q = 0,$$

whereas the final order that we will account for implies the equation

$$(5.16) \qquad 2p\,(\partial_{x_1}\phi_2) + 2q\,(\partial_{x_2}\phi_2) + \partial_{x_1}^2\phi_1 + \partial_{x_2}^2\phi_1 + (\partial_{x_1}\phi_1)^2 + (\partial_{x_2}\phi_1)^2 = 0.$$

Amplitude expansion. It is convenient to remove the singularities from ϕ_1 and ϕ_2. This is accomplished by the change of functions,

$$(5.17) \qquad \phi_1 = -\tfrac{1}{2}\log\phi + \psi_1,$$
$$(5.18) \qquad \phi_2 = \tfrac{1}{8}\phi^{-1} + \psi_2.$$

With this change, (5.15)–(5.16) take the form

$$(5.19) \qquad 2p\,(\partial_{x_1}\psi_1) + 2q\,(\partial_{x_2}\psi_1) + \phi\,[\partial_{x_1}^2 + \partial_{x_2}^2]\log\phi = 0,$$
$$(5.20) \qquad 2p\,(\partial_{x_1}\psi_2) + 2q\,(\partial_{x_2}\psi_2) + \partial_{x_1}^2\psi_1 + \partial_{x_2}^2\psi_1 + (\partial_{x_1}\psi_1)^2 + (\partial_{x_2}\psi_1)^2 = 0,$$

supplemented with the initial conditions $\psi_1 = \psi_2 = 0$ at $x_{1,2} = x_{1,2}'$.

Expansion in $z = \mathcal{O}(k_0^{-1})$. We now make the assumption that the propagation distance satisfies

$$(5.21) \qquad k_0 \underbrace{|x_3 - x_3'|}_{y_3 = z} = \mathcal{O}(1).$$

Thus we guarantee that the stationary point (where $\partial_\zeta \phi = 0$) of the integral representation (3.7) remains at $\zeta = 0$, and that

$$(5.22) \qquad |\exp[\mathrm{i}k_0 |x_3 - x_3'|\zeta]| = \mathcal{O}(1).$$

We then expand the relevant functions about $\zeta = 0$, i.e.,

$$(5.23) \qquad \phi = I_0 - \tfrac{1}{2}\zeta^2 I_1 - \tfrac{1}{8}\zeta^4 I_2 + \dots,$$

$$(5.24) \qquad \psi_1 = \psi_{10} + \zeta^2 \psi_{11} + \dots,$$

$$(5.25) \qquad \psi_2 = \psi_{20} + \dots,$$

where I_0, I_1, I_2 and $\psi_{10}, \psi_{11}, \psi_{20}$ are independent of ζ.

The phase function. Invoking expansion (5.23) into (5.14), the equations determining the phase function become

$$(5.26) \qquad P^2 + Q^2 - n^2(x_{1,2}) = 0,$$

$$(5.27) \qquad P\,(\partial_{x_1} I_1) + Q\,(\partial_{x_2} I_1) - 1 = 0,$$

$$(5.28) \qquad P\,(\partial_{x_1} I_2) + Q\,(\partial_{x_2} I_2) - (\partial_{x_1} I_1)^2 - (\partial_{x_2} I_1)^2 = 0,$$

where $P = \partial_{x_1} I_0$ and $Q = \partial_{x_2} I_0$. With eikonal equation (5.26) is associated the Hamilton system

$$(5.29) \qquad \frac{\mathrm{d}x_1}{\mathrm{d}\mu} = P, \qquad \frac{\mathrm{d}P}{\mathrm{d}\mu} = (\partial_1 M)(x_{1,2}),$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}\mu} = Q, \qquad \frac{\mathrm{d}Q}{\mathrm{d}\mu} = (\partial_2 M)(x_{1,2}),$$

where $M = \tfrac{1}{2}n^2$, supplemented by the initial conditions

$$(5.30) \qquad (x_1, x_2)|_0 = (x_1', x_2'), \quad (P, Q)|_0 = (\alpha_1, \alpha_2), \quad \alpha_1^2 + \alpha_2^2 = n^2(x_{1,2}').$$

The additional equations (5.27)–(5.28) comply with the initial conditions at $\mu = 0$: $I_j = 0$, $j = 0, 1, 2, \dots$.

In the Hamilton system (5.29), we expand the right-hand sides into a Taylor series about the "source" coordinates, $x_{1,2}'$:

$$(5.31) \qquad \frac{\mathrm{d}P}{\mathrm{d}\mu} = (\partial_1 M)(x_{1,2}') + y_1 \partial_1(\partial_1 M)(x_{1,2}') + y_2 \partial_2(\partial_1 M)(x_{1,2}'),$$

$$\frac{\mathrm{d}Q}{\mathrm{d}\mu} = (\partial_2 M)(x_{1,2}') + y_1 \partial_1(\partial_2 M)(x_{1,2}') + y_2 \partial_2(\partial_2 M)(x_{1,2}').$$

We then evaluate the solutions to the Hamilton (see (5.29)–(5.31)) and eikonal (see (5.26)) equations for small values of μ. The parametric representation of the Hamil-

tonian flow follows as

$$y_1 = P|_0 \mu + \tfrac{1}{2} P_1 \mu^2 + \tfrac{1}{3} P_2 \mu^3 + \dots,$$

$$P = \underbrace{\alpha_1}_{P|_0} + \underbrace{(\partial_1 M)}_{P_1} \mu + \tfrac{1}{2} \underbrace{[\alpha_1 \partial_1 + \alpha_2 \partial_2](\partial_1 M)}_{P_2} \mu^2 + \dots,$$

(5.32)

$$y_2 = Q|_0 \mu + \tfrac{1}{2} Q_1 \mu^2 + \tfrac{1}{3} Q_2 \mu^3 + \dots,$$

$$Q = \underbrace{\alpha_2}_{Q|_0} + \underbrace{(\partial_2 M)}_{Q_1} \mu + \tfrac{1}{2} \underbrace{[\alpha_1 \partial_1 + \alpha_2 \partial_2](\partial_2 M)}_{Q_2} \mu^2 + \dots;$$

in these equations we differentiate $(\partial_{1,2})$ with respect to $x'_{1,2}$ while the argument of $(\partial_{1,2} M)$ is $x'_{1,2}$. For the purpose of the uniform matching, we will re-expand the solution about the transverse midpoint coordinates $\bar{x}_{1,2}$ and give results as needed later.

Solving system (5.32) for μ, α_1, α_2 in terms of y_1, y_2, yields

$$\mu = \frac{r_2}{n} \left(1 - \frac{1}{2} \frac{[y_1 \partial_1 + y_2 \partial_2]n}{n} + \frac{1}{3} \left(\frac{[y_1 \partial_1 + y_2 \partial_2]n}{n} \right)^2 \right.$$

(5.33)

$$\left. + \frac{1}{8} \left(\frac{[y_1^\perp \partial_1 + y_2^\perp \partial_2]n}{n} \right)^2 - \frac{1}{6} \frac{[y_1 \partial_1 + y_2 \partial_2]^2 n}{n} \right),$$

(5.34) $$\alpha_{1,2} = \frac{n}{r_2} \left(y_{1,2}(1 - \tfrac{1}{2} a_1^2) + y_{1,2}^\perp (a_1 + a_2) \right),$$

where the argument of n is $x'_{1,2}$,

(5.35) $$r_2 = (y_1^2 + y_2^2)^{1/2}$$

and

(5.36) $$y_1^\perp = -y_2, \quad y_2^\perp = y_1,$$

while

$$a_1 = -\frac{[y_1^\perp \partial_1 + y_2^\perp \partial_2]n}{2n} \ ,$$

(5.37)

$$a_2 = \frac{1}{12} \left(\frac{([y_1^\perp \partial_1 + y_2^\perp \partial_2]n)([y_1 \partial_1 + y_2 \partial_2]n)}{n^2} \right.$$

$$\left. -2 \frac{[y_1^\perp \partial_1 + y_2^\perp \partial_2][y_1 \partial_1 + y_2 \partial_2]n}{n} \right);$$

note that a_1 and a_2 are of first and second order in y, respectively.

With the aid of relation

$$P \, \partial_{x_1} I_j + Q \, \partial_{x_2} I_j = \frac{\mathrm{d} I_j}{\mathrm{d} \mu}$$

valid along a characteristic or ray, (5.26)–(5.28) take the form

$$(5.38) \qquad \frac{\mathrm{d}I_0}{\mathrm{d}\mu} = n^2(x_{1,2}),$$

$$(5.39) \qquad \frac{\mathrm{d}I_1}{\mathrm{d}\mu} = 1,$$

$$(5.40) \qquad \frac{\mathrm{d}I_2}{\mathrm{d}\mu} = (\partial_{x_1} I_1)^2 + (\partial_{x_2} I_1)^2.$$

Explicit expansions of I_j near the "source" (μ small or, equivalently, r_2 small) are readily obtained from (5.38)–(5.40) using system (5.32) and solutions (5.33)–(5.34). (Basically, such a procedure encompasses an expansion of (the argument of) n^2 about the fixed initial point $(x'_{1,2})$ in terms of $y_{1,2}$; we then substitute the small μ expansion (5.32) for $y_{1,2}$ and re-expand the relevant coefficients about $\bar{x}_{1,2}$.) These expansions are only needed for matching the inner and outer solutions. They are given as needed later (see (5.54)–(5.58)).

The amplitude expansion. Invoking expansion (5.23)–(5.25) into (5.19)–(5.20), the equations determining the amplitude become

$$(5.41) \quad 2P\,\partial_{x_1}\psi_{10} + 2Q\,\partial_{x_2}\psi_{10} + I_0[\partial_{x_1}^2 + \partial_{x_2}^2]\log I_0 = 0,$$

$$(5.42) \quad 2P\,\partial_{x_1}\psi_{20} + 2Q\,\partial_{x_2}\psi_{20} + (\partial_{x_1}\psi_{10})^2 + (\partial_{x_2}\psi_{10})^2 + [\partial_{x_1}^2 + \partial_{x_2}^2]\psi_{10} = 0,$$

supplemented with the initial conditions

$$(5.43) \qquad \psi_{10} = \psi_{20} = 0 \quad \text{at } x_{1,2} = x'_{1,2}.$$

The next order equation, for ψ_{11}, becomes

$$2P\,\partial_{x_1}\psi_{11} + 2Q\,\partial_{x_2}\psi_{11} - (\partial_{x_1}I_1)(\partial_{x_1}\psi_{10}) - (\partial_{x_2}I_1)(\partial_{x_2}\psi_{10})$$

$$(5.44) \qquad - \tfrac{1}{2}I_1[\partial_{x_1}^2 + \partial_{x_2}^2]\log I_0 - \tfrac{1}{2}I_0[\partial_{x_1}^2 + \partial_{x_2}^2](I_1/I_0) = 0,$$

supplemented with the initial conditions

$$(5.45) \qquad \psi_{11} = 0 \quad \text{at } x_{1,2} = x'_{1,2}.$$

In (5.41)–(5.42) and (5.44),

$$P\,\partial_{x_1}\psi_{ij} + Q\,\partial_{x_2}\psi_{ij} = \frac{\mathrm{d}\psi_{ij}}{\mathrm{d}\mu}$$

along a characteristic or ray. Upon solving these equations, about the stationary point at $\zeta = 0$, we obtain the transform-domain expansion for the characteristic Green's function,

$$\tilde{G}(x_{1,2}, x'_{1,2}; \zeta)\,\exp[-\mathrm{i}k_0(x_3 - x'_3)\zeta] = \mathsf{C}\,\frac{1}{\sqrt{I_0}}\,\exp[\mathrm{i}k_0(I_0 - \tfrac{1}{2}\zeta^2 I_1) + \psi_{10}]$$

$$\left\{ 1 - \mathrm{i}k_0\zeta(x_3 - x'_3) + \frac{1}{\mathrm{i}k_0}\left(\frac{1}{8I_0} + \psi_{20} + \mathrm{i}k_0\zeta^2\left(\psi_{11} + \frac{I_1}{4I_0} + \tfrac{1}{2}[\mathrm{i}k_0(x_3 - x'_3)]^2 \right) \right.\right.$$

$$(5.46) \qquad \left.\left. - \tfrac{1}{8}(\mathrm{i}k_0\zeta^2)^2 I_2 \right) + \dots \right\}.$$

TABLE 5.1
Relevant equations.

I_0	I_1	I_2	ψ_{10}	ψ_{20}	ψ_{11}
(5.38)	(5.39)	(5.40)	(5.41)	(5.42)	(5.44)

Carrying out the inverse Fourier transform with the method of stationary phase then results in

$$G(x_{1,2}, x_3 - x_3'; x_{1,2}') = \mathsf{C} \left(\frac{-\mathrm{i}k_0}{2\pi} \right)^{1/2} \exp(\psi_{10}) \frac{\exp(\mathrm{i}k_0 I_0)}{(I_0 I_1)^{1/2}}$$

$$(5.47) \quad \left\{ 1 + \frac{1}{\mathrm{i}k_0 I_0} \left(\frac{3}{8} \left(1 - \frac{I_2 I_0}{I_1^2} \right) + \left(\frac{\psi_{11}}{I_1} + \psi_{20} \right) I_0 + \tfrac{1}{2} [\mathrm{i}k_0 (x_3 - x_3')]^2 \frac{I_0}{I_1} \right) + \dots \right\}.$$

Effective index of refraction, effective metric and uniform asymptotic expansion. As in the two-dimensional case [10], for notational convenience, we introduce the *effective* index of refraction and *effective* horizontal distance as

$$(5.48) \qquad\qquad \nu \equiv \left[\frac{I_0}{I_1} \right]^{1/2},$$

$$(5.49) \qquad\qquad \chi_1 \equiv [I_0 I_1]^{1/2},$$

where the arguments are evaluated along the characteristics, whereas

$$(5.50) \qquad\qquad r = [\chi_1^2 + z^2]^{1/2}.$$

Then a uniform asymptotic expansion is

$$(5.51) \quad G(x_{1,2}, x_3 - x_3'; x_{1,2}') = \frac{1}{4\pi r} \exp(\mathrm{i}k_0 \nu r) \exp(\psi_{10} r^2 / \chi_1^2)$$

$$\left\{ 1 + \frac{r}{\mathrm{i}k_0 \nu \chi_1^2} \left(\frac{3}{8} \left(1 - \frac{\nu^3 I_2}{\chi_1} \right) + \nu(\nu \psi_{11} + \chi_1 \psi_{20}) \right) \right.$$

$$\left. + \frac{1}{8} \left(1 - \frac{\nu^3 I_2}{\chi_1} \right) \left(\frac{(x_3 - x_3')^2}{\chi_1^2 r^2} [\mathrm{i}k_0 \nu r (x_3 - x_3')^2 + r^2 + \chi_1^2] \right) + \dots \right\}.$$

The equations to be evaluated or solved are listed in Table 5.1.

In the *outer* region, $\chi_1 = \mathcal{O}(1)$, whence

$$(5.52) \qquad\qquad \frac{r}{\chi_1} \sim 1 + \frac{(x_3 - x_3')^2}{2\chi_1^2} = 1 + \mathcal{O}(k_0^{-2}),$$

$$(5.53) \qquad\qquad \nu r = \nu \chi_1 + \nu \frac{(x_3 - x_3')^2}{2\chi_1} + \mathcal{O}(k_0^{-4}),$$

and the uniform solution reduces to the outer solution (5.47) with

$$\mathsf{C} = \left(\frac{\mathrm{i}}{8\pi k_0} \right)^{1/2}.$$

On the *inner* region, $\chi_1 = \mathcal{O}(k_0^{-1})$, whence

$$(5.54) \quad \frac{1}{\chi_1^2}\left(1 - \frac{\nu^3 I_2}{\chi_1}\right) \sim -\frac{1}{3}\left[\left(\frac{\partial_1 n}{n}\right)^2 + \left(\frac{\partial_2 n}{n}\right)^2\right],$$

$$(5.55)\,\frac{\nu}{\chi_1^2}(\nu\psi_{11} + \chi_1\psi_{20}) \sim \frac{1}{12}\left[\frac{[\partial_1^2 + \partial_2^2]n}{n} + \left(\frac{\partial_1 n}{n}\right)^2 + \left(\frac{\partial_2 n}{n}\right)^2\right],$$

$$(5.56) \qquad\qquad \frac{\psi_{10}}{\chi_1^2} \sim \frac{1}{12}\left[-\frac{[\partial_1^2 + \partial_2^2]n}{n} + \left(\frac{\partial_1 n}{n}\right)^2 + \left(\frac{\partial_2 n}{n}\right)^2\right],$$

$$\nu r \sim ny\left(1 + \frac{1}{24}\left\{\frac{[y_1\partial_1 + y_2\partial_2]^2 n}{n} + \left(\frac{y_1\partial_1 n}{n}\right)^2 + \left(\frac{y_2\partial_2 n}{n}\right)^2\right.\right.$$

$$(5.57) \qquad\qquad \left.\left. + \left[\left(\frac{\partial_1 n}{n}\right)^2 + \left(\frac{\partial_2 n}{n}\right)^2\right]\left(-y^2 + \frac{(x_3 - x_3')^4}{y^2}\right)\right\}\right),$$

$$(5.58) \qquad\qquad \frac{1}{r} \sim \frac{1}{y}\left(1 - \frac{1}{24}\left[\left(\frac{\partial_1 n}{n}\right)^2 + \left(\frac{\partial_2 n}{n}\right)^2\right]\frac{r_2^4}{y^2}\right),$$

where r_2 was defined in (5.35), and the argument of n is $\bar{x}_{1,2}$. Substitution of these results into the uniform solution yields the inner solution (5.10).

The inner and outer solutions match when $\chi_1 = \mathcal{O}(k_0^{-2/3})$, which scaling defines the boundary layer.

6. Uniform asymptotic expansion of the characteristic Green's function: The presence of a caustic. In the generic case of a heterogeneous slab, caustics will form in the transverse directions. Following the Maslov approach, we note that there will always be two coordinates chosen from (y_1, y_2) and their Fourier duals (η_1, η_2) such that the solution in these coordinates remains asymptotically finite and meaningful. The transition from one doublet of coordinates to another is followed by the Keller–Maslov line bundle [11] that is accounted for in the solution's amplitude through a tensor product. We will discuss the mixed (y_1, η_2) case here; together with the previous section, all necessary combinations can be found by permutation of coordinates.

We reconsider (3.4) once again,

$$[\partial_k \partial_k + k_0^2 n^2(x_{1,2})]\, G(x_{1,2}, x_3 - x_3'; x_{1,2}') = -\delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3'),$$

and introduce a slight change in notation,

$$y_{1,2} = x_{1,2} - x_{1,2}', \qquad z = x_3 - x_3'.$$

We write the Green's function in the form of an appropriate Fourier integral,

$$(6.1) \qquad G(x_{1,2}, x_3 - x_3'; x_{1,2}') = \frac{k_0}{2\pi}\int_{\mathbb{R}} \hat{G}(y_1, \eta_2, z; x_{1,2}')\,\exp(\mathrm{i}k_0\eta_2 y_2)\,\mathrm{d}\eta_2.$$

We now distinguish amplitude and phase according to

$$(6.2) \qquad \hat{G}(y_1, \eta_2, z; x_{1,2}') = A(y_1, \eta_2, z; x_{1,2})\exp[\mathrm{i}k_0\phi(y_1, \eta_2, z; x_{1,2}')].$$

The inner solution. We expand the index of refraction, $n^2(x'_{1,2} + y_{1,2})$, in the partial differential equation in y_2 about $-q$ where $q \equiv \partial_{\eta_2}\phi$. Upon substituting the Fourier representation into the Helmholtz equation (3.4), after several integrations by parts, we obtain the following equations.

Up to highest order (lowest order in $(ik_0)^{-1}$), we recover the eikonal equation, viz.,

$$(6.3) \qquad p^2 + \eta_2^2 + r^2 - n^2(x'_1 + y_1, x'_2 - q) = 0,$$

where $p \equiv \partial_{y_1}\phi$ and $r \equiv \partial_z\phi$. This is a pseudodifferential equation for ϕ. In the spirit of the solution method of characteristics, we deduce the Hamilton equations for the bicharacteristics,

$$\frac{\mathrm{d}y_1}{\mathrm{d}\lambda} = p, \qquad\qquad \frac{\mathrm{d}p}{\mathrm{d}\lambda} = (\partial_1 M)(x'_1 + y_1, x'_2 - q),$$

$$(6.4) \qquad \frac{\mathrm{d}\eta_2}{\mathrm{d}\lambda} = (\partial_2 M)(x'_1 + y_1, x'_2 - q), \qquad \frac{\mathrm{d}q}{\mathrm{d}\lambda} = -\eta_2,$$

$$\frac{\mathrm{d}z}{\mathrm{d}\lambda} = r, \qquad\qquad \frac{\mathrm{d}r}{\mathrm{d}\lambda} = 0,$$

in which

$$M = \tfrac{1}{2}n^2.$$

The Hamilton system is supplemented with initial conditions at $\lambda = 0$:

$$(6.5) \qquad (y_1, \eta_2, z)|_0 = (0, \beta_2, 0), \quad (p, q, r)|_0 = (\beta_1, 0, \beta_3), \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = n^2(x'_{1,2}).$$

In the next order, we recover the transport-like equation for A, viz.,

$$(6.6) \qquad (\partial_{y_1}^2 + \partial_z^2)A + 2ik_0 CA - DA = -\delta(y_1)\delta(z),$$

in which

$$CA = [p\,\partial_{y_1} + r\partial_z + (\partial_2 M)(x'_1 + y_1, x'_2 - q)\,\partial_{\eta_2}]\,A$$
$$(6.7) \qquad + \tfrac{1}{2}\left[\partial_{y_1}^2\phi + \partial_z^2\phi - (\partial_2^2 M)(x'_1 + y_1, x'_2 - q)\,\partial_{\eta_2}^2\phi\right]A,$$
$$DA = \left[(\partial_2^2 M)(x'_1 + y_1, x'_2 - q)\,\partial_{\eta_2}^2 - (\partial_{\eta_2}q)(\partial_2^3 M)(x'_1 + y_1, x'_2 - q)\,\partial_{\eta_2}\right]A$$
$$\qquad + \left[-\tfrac{1}{3}(\partial_{\eta_2}^2 q)(\partial_2^3 M)(x'_1 + y_1, x'_2 - q) + \tfrac{1}{4}(\partial_{\eta_2}q)^2(\partial_2^4 M)(x'_1 + y_1, x'_2 - q)\right]A$$
$$(6.8) \qquad + \mathcal{O}((ik_0)^{-1}).$$

Observe that on the inner region the y_1 and z derivatives are large, and, hence, the inner transport-like equation reduces to

$$(6.9) \qquad (\partial_{y_1}^2 + \partial_z^2)A + 2ik_0 CA = -\delta(y_1)\delta(z) + \ldots.$$

The phase function. First, we evaluate the solutions to the Hamilton (cf. (6.4)) and eikonal (cf. (6.3)) equations for small values of λ. The parametric representation of the Hamiltonian flow follows from (6.4) as

$$y_1 = \beta_1\lambda + \tfrac{1}{2}(\partial_1 M)\,\lambda^2 + \ldots, \qquad p = \beta_1 + (\partial_1 M)\,\lambda + \ldots,$$

$$(6.10) \qquad \eta_2 = \beta_2 + (\partial_2 M)\,\lambda + \ldots, \qquad q = -\beta_2\lambda - \tfrac{1}{2}(\partial_2 M)\,\lambda^2 + \ldots,$$

$$z = \beta_3\lambda, \qquad\qquad r = \beta_3$$

while, through integration of the canonical one-form along the bicharacteristic, the phase function is found to be

$$(6.11) \qquad \phi = (\beta_1^2 + \beta_3^2)\,\lambda + [\beta_1(\partial_1 M) - \tfrac{1}{2}\beta_2(\partial_2 M)]\,\lambda^2 + \dots,$$

in which the substitution $(\partial_{1,2}M) = (\partial_{1,2}M)(x'_{1,2})$ is understood.

Solving system (6.10) subject to the constraint in (6.5) for $\lambda, \beta_1, \beta_2, \beta_3$ in terms of y_1, η_2, z yields

$$(6.12) \qquad \lambda = (R/\gamma)[1 - \gamma^{-3}(\tfrac{1}{2}\gamma y_1(\partial_1 M) + \eta_2 R(\partial_2 M)) + \dots],$$

$$(6.13) \qquad \beta_1 = (\gamma/R)[y_1 + \gamma^{-3}(-\tfrac{1}{2}\gamma z^2(\partial_1 M) + \eta_2 y_1 R(\partial_2 M)) + \dots],$$

$$(6.14) \qquad \beta_2 = \eta_2 - (R/\gamma)(\partial_2 M) + \dots,$$

$$(6.15) \qquad \beta_3 = (\gamma z/R)[1 + \gamma^{-3}(\tfrac{1}{2}\gamma y_1(\partial_1 M) + \eta_2 R(\partial_2 M)) + \dots],$$

in which $R \equiv (y_1^2 + z^2)^{1/2}$ and $\gamma \equiv [n^2(x'_{1,2}) - \eta_2^2]^{1/2}$. Substituting these solutions in the remaining equations of system (6.10) gives

$$p = (\gamma/R)[y_1 + \gamma^{-3}(\tfrac{1}{2}\gamma(y_1^2 + R^2)(\partial_1 M) + \eta_2 y_1 R(\partial_2 M)) + \dots],$$

$$(6.16) \qquad q = (R/\gamma)[-\eta_2 + \gamma^{-4}\tfrac{1}{2}(\gamma^2 y_1 \eta_2(\partial_1 M) + (\gamma^2 + 2\eta_2^2)R(\partial_2 M)) + \dots],$$

$$r = (\gamma z/R)[1 + \gamma^{-3}(\tfrac{1}{2}\gamma y_1(\partial_1 M) + \eta_2 R(\partial_2 M)) + \dots],$$

whereas the phase function (6.11) takes the form

$$(6.17) \qquad \phi = \gamma R\,[1 + \gamma^{-3}(\tfrac{1}{2}\gamma y_1(\partial_1 M) + \tfrac{1}{2}\eta_2 R(\partial_2 M)) + \dots].$$

The amplitude function. Having obtained the solution of the eikonal equation, we now proceed with solving the transport-like equation. First, observe the following property of functions F of $k_0\phi$:

$$\left[\partial_{y_1}^2 + \partial_z^2 + 2ik_0 C\right] F = k_0^2(p^2 + r^2)$$

$$\left\{ F'' + 2iF' + \frac{1}{k_0\phi}\left(F' + iF\right) + \frac{\eta_2(\partial_2 M)(x'_{1,2})}{2k_0\gamma^4}\left(F' + iF - 4ik_0\phi F'\right) + \dots \right\}.$$

Using this property, the inner solution of (6.9) is constructed and found to be

$$(6.18) \quad A = \frac{i}{4}\exp(-ik_0\phi)$$
$$\left\{ H_0^{(1)}(k_0\phi) + \frac{\eta_2(\partial_2 M)(x'_{1,2})}{2\gamma^4}k_0\phi^2[H_1^{(1)}(k_0\phi) - iH_2^{(1)}(k_0\phi)] + \dots \right\}.$$

The outer solution. We assume that our wave field is a transient phenomenon with dominant wave number k_0. The outer region is determined by the condition

$$\|(x_1 - x'_1, x_2 - x'_2)\| = \mathcal{O}(1)$$

and corresponds to the behavior of the kernels away from their diagonals.

Amplitude expansion. In the outer region the derivatives of the amplitude A in (6.2) are $\mathcal{O}(1)$. Thus we expand

$$(6.19) \qquad A = \frac{1}{(k_0\phi)^{1/2}}\left\{ A_0 + \frac{1}{ik_0}\left(\frac{A_0}{8\phi} + A_1\right) + \dots \right\}.$$

With this definition, A_0 and A_1 are continuous near the "source" (at $x'_{1,2}$). Substitution of (6.19) into (6.6) and setting terms proportional to $k_0^{-n+1/2}$, $n = 0, 1, \ldots$, equal to zero yields the transport equations

$$(6.20) \qquad LA_0 = 0,$$

$$
\begin{aligned}
LA_1 - \phi^{1/2} D(\phi^{-1/2} A_0) + \partial_{y_1}^2 A_0 + \partial_z^2 A_0 \\
+ \frac{1}{\phi}(\partial_2 M)(x'_1 + y_1, x'_2 - q)\left(\partial_{\eta_2} A_0 + \frac{3}{4}\frac{q}{\phi} A_0\right) \\
(6.21) \qquad - \tfrac{1}{2}(\partial_{\eta_2} q)(\partial_2^2 M)(x'_1 + y_1, x'_2 - q)A_0 = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
LA = 2\frac{\mathrm{d}A}{\mathrm{d}\lambda} + \bigg\{ \phi\,[\partial_{y_1}^2 + \partial_z^2]\log\phi \\
(6.22) \qquad - \frac{q}{\phi}(\partial_2 M)(x'_1 + y_1, x'_2 - q) - (\partial_{\eta_2} q)(\partial_2^2 M)(x'_1 + y_1, x'_2 - q) \bigg\} A.
\end{aligned}
$$

The nonhomogeneous terms in (6.21) are continuous near the "source."

In preparation for matching the inner and outer solutions, we consider the small λ expansion of the solutions to (6.20)–(6.21):

$$
\begin{aligned}
A_0 \to A_0|_0 \left(1 - \frac{3\beta_2\lambda(\partial_2 M)(x'_{1,2})}{4(\beta_1^2 + \beta_3^2)} + \ldots\right) \\
(6.23) \qquad = A_0|_0 \left(1 - \frac{3\beta_2(\partial_2 M)(x'_{1,2})\phi}{4(\beta_1^2 + \beta_3^2)^2} + \ldots\right),
\end{aligned}
$$

$$(6.24) \qquad A_1 \to A_1|_0 + \ldots.$$

Thus near the "source," (6.19) takes the form

$$
A = \left(\frac{1}{k_0\phi}\right)^{1/2}\left\{A_0|_0\left(1 + \frac{1}{8ik_0\phi} - \frac{3\beta_2(\partial_2 M)(x'_{1,2})}{32(\beta_1^2 + \beta_3^2)^2}\left[8\phi + \frac{1}{ik_0}\right]\right) + \frac{1}{ik_0}A_1|_0 + \ldots\right\}.
$$
$$(6.25)$$

Inner solution on the outer scale. In preparation for developing the inner solution on the outer region, we observe the asymptotic behavior of the amplitude given in (6.18): We have

$$(6.26) \quad A = \left(\frac{i}{8\pi k_0\phi}\right)^{1/2}\left\{1 + \frac{1}{8ik_0\phi} - \frac{3\eta_2(\partial_2 M)(x'_{1,2})}{4\gamma^4}\left[\phi - \frac{5}{8ik_0}\right] + \ldots\right\}$$

as $k_0\phi$ becomes large. On the other hand, approaching the "source" as λ (i.e., ϕ) becomes small, in this expression, gives

$$
\frac{\eta_2}{\gamma^4} \to \frac{\beta_2}{(\beta_1^2 + \beta_3^2)^2},
$$

cf. (6.10).

Uniform asymptotic expansion. *Matching the inner solution on the overlapping region.* The overlapping region is governed by λ small and $k_0\lambda$ large, i.e., ϕ small and $k_0\phi$ large (or $R = \mathcal{O}(k_0^{-1/3})$, where R was defined just below (6.15)). Comparing (6.25) with (6.26) yields the initial conditions for A_0 and A_1,

$$(6.27) \qquad A_0|_0 = \left(\frac{\mathrm{i}}{8\pi}\right)^{1/2},$$

$$(6.28) \qquad A_1|_0 = \left(\frac{\mathrm{i}}{8\pi}\right)^{1/2} \frac{9\beta_2(\partial_2 M)(x'_{1,2})}{16(\beta_1^2 + \beta_3^2)^2}.$$

Thus the initial conditions for A_0 and A_1 are determined by matching the outer expansion to the inner solution on the overlapping region. It is only now that the outer solution is fully determined.

Uniform expansion. Finally, the uniform expansion is obtained by adding the outer solution ((6.1), (6.2), and (6.19)) to the inner solution ((6.1), (6.2), and (6.18)) and subtracting the matching terms on the overlapping region (equation (6.26)). However, in the inner region caustics will not have developed yet and, hence, there the noncaustic uniform asymptotic expansion of the previous section will apply.

Expansion in $z = \mathcal{O}(k_0^{-1})$. For use of the expansion of G in the kernels of the vertical slowness operator and the thin-slab propagator, we will have to make the assumption that the propagation distance satisfies

$$(6.29) \qquad k_0 \underbrace{|x_3 - x'_3|}_{z} = \mathcal{O}(1).$$

Exploiting the small range of propagation to yield the thin-slab propagator, we expand

$$(6.30) \qquad \phi = I_0 + \tfrac{1}{2}z^2 I_1 + \tfrac{1}{8}z^4 I_2 + \dots,$$

$$(6.31) \qquad A_0 = A_{00} + z^2 A_{01} + \dots,$$

$$(6.32) \qquad A_1 = A_{10} + \dots,$$

where I_0, I_1, I_2 and A_{00}, A_{01}, A_{10} are independent of z.

The phase function. Substituting the expansion for ϕ in (6.3) yields up to leading order

$$(6.33) \qquad P^2 + \eta_2^2 - n^2(x'_1 + y_1, x'_2 - Q) = 0,$$

where $P = \partial_{y_1} I_0$ and $Q = \partial_{\eta_2} I_0$. The associated Hamilton system, i.e., the counterpart of (6.4) with the preferred (principal) components removed, becomes

$$(6.34) \qquad \begin{aligned} &\frac{\mathrm{d}y_1}{\mathrm{d}\mu} = P, & &\frac{\mathrm{d}P}{\mathrm{d}\mu} = (\partial_1 M)(x'_1 + y_1, x'_2 - Q), \\ &\frac{\mathrm{d}\eta_2}{\mathrm{d}\mu} = (\partial_2 M)(x'_1 + y_1, x'_2 - Q), & &\frac{\mathrm{d}Q}{\mathrm{d}\mu} = -\eta_2, \end{aligned}$$

supplemented by the initial conditions (cf. (6.5))

$$(6.35) \qquad (y_1, \eta_2)|_0 = (0, \alpha_2), \quad (P, Q)|_0 = (\alpha_1, 0), \quad \alpha_1^2 + \alpha_2^2 = n^2(x'_{1,2}).$$

The equation for the next order term follows as

$$\frac{\mathrm{d}I_1}{\mathrm{d}\mu} + I_1^2 = 0 \tag{6.36}$$

with solution

$$I_1 = \frac{1}{\mu}. \tag{6.37}$$

(The initial condition has been matched with the inner solution.)

The equation for I_2 follows as

$$\frac{\mathrm{d}I_2}{\mathrm{d}\mu} + 4I_2 I_1 + (\partial_{y_1} I_1)^2 - (\partial_2^2 M)(x_1' + y_1, x_2' - Q)(\partial_{\eta_2} I_1)^2 = 0. \tag{6.38}$$

This equation simplifies from a computational point of view upon scaling $I_2 = \mu^{-4}\bar{I}_2$; then

$$\frac{\mathrm{d}\bar{I}_2}{\mathrm{d}\mu} + [(\partial_{y_1} I_1)^2 - (\partial_2^2 M)(x_1' + y_1, x_2' - Q)(\partial_{\eta_2} I_1)^2]\mu^4 = 0, \tag{6.39}$$

supplemented by the initial condition

$$\bar{I}_2|_0 = 0. \tag{6.40}$$

We evaluate the solutions to the Hamilton (equation (6.34)) and eikonal (equation (6.33)) equations for small values of μ. The parametric representation of the Hamiltonian flow follows as (compare with (6.10))

$$\begin{aligned}
&y_1 = \alpha_1\mu + \tfrac{1}{2}(\partial_1 M)\,\mu^2 + \ldots, \qquad P = \alpha_1 + (\partial_1 M)\,\mu + \ldots, \\
&\eta_2 = \alpha_2 + (\partial_2 M)\,\mu + \ldots, \qquad\quad Q = -\alpha_2\mu - \tfrac{1}{2}(\partial_2 M)\,\mu^2 + \ldots,
\end{aligned} \tag{6.41}$$

while the leading-order constituent phase function is found to be (compare with (6.11))

$$I_0 = \alpha_1^2\mu + [\alpha_1(\partial_1 M) - \tfrac{1}{2}\alpha_2(\partial_2 M)]\,\mu^2 + \ldots, \tag{6.42}$$

in which the substitution $(\partial_{1,2}M) = (\partial_{1,2}M)(x_{1,2}')$ is understood.

Solving system (6.41) subject to the constraint in (6.35) for μ, α_1, α_2 in terms of y_1, η_2 yields

$$\mu = (|y_1|/\gamma)[1 - \gamma^{-3}(\tfrac{1}{2}\gamma y_1(\partial_1 M) + \eta_2|y_1|(\partial_2 M)) + \ldots], \tag{6.43}$$

$$\alpha_1 = (\gamma/|y_1|)[y_1 + \gamma^{-3}\eta_2 y_1^2(\partial_2 M) + \ldots], \tag{6.44}$$

$$\alpha_2 = \eta_2 - (|y_1|/\gamma)(\partial_2 M) + \ldots, \tag{6.45}$$

in which $\gamma = [n^2(x_{1,2}') - \eta_2^2]^{1/2}$ as before. Substituting these solutions into (6.42) then yields (compare with (6.17))

$$I_0 = \gamma|y_1|[1 + \gamma^{-3}(\tfrac{1}{2}\gamma y_1(\partial_1 M) + \tfrac{1}{2}\eta_2|y_1|(\partial_2 M)) + \ldots]. \tag{6.46}$$

Amplitude expansion. Upon substituting (6.30)–(6.32) into (6.20)–(6.21), and collecting leading-order terms, the equations for A_{00} and A_{10} follow as

$$\widetilde{L}A_{00} = 0, \tag{6.47}$$

$$\widetilde{L}A_{10} + 2A_{01} - \frac{3}{4}\frac{Q}{I_0^2}(\partial_2 M)(x_1' + y_1, x_2' - Q)\,A_{00} - \widetilde{D}A_{00} = 0, \tag{6.48}$$

where

$$\widetilde{L}A = 2\frac{\mathrm{d}A}{\mathrm{d}\mu} + \left[I_0\,\partial_{y_1}^2 \log I_0 + I_1 - \frac{Q}{I_0}(\partial_2 M)(x_1' + y_1, x_2' - Q) \right.$$

(6.49)
$$\left. - (\partial_{\eta_2}Q)(\partial_2^2 M)(x_1' + y_1, x_2' - Q) \right] A,$$

$$\widetilde{D}A = -\partial_{y_1}^2 A - \frac{1}{I_0}(\partial_2 M)(x_1' + y_1, x_2' - Q)\,\partial_{\eta_2} A$$

$$+ I_0^{1/2} \left[(\partial_2^2 M)(x_1' + y_1, x_2' - Q)\left(\partial_{\eta_2}^2 + \frac{\partial_{\eta_2}Q}{2I_0}\right) \right.$$

$$- (\partial_2^3 M)(x_1' + y_1, x_2' - Q)((\partial_{\eta_2}Q)\partial_{\eta_2} + \tfrac{1}{3}(\partial_{\eta_2}^3 Q))$$

(6.50)
$$\left. + \tfrac{1}{4}(\partial_2^4 M)(x_1' + y_1, x_2' - Q)\tfrac{1}{4}(\partial_{\eta_2}Q)^2 \right] \left(\frac{A}{I_0^{1/2}} \right),$$

supplemented with the initial conditions (compatible with (6.27)–(6.28))

(6.51)
$$A_{00}|_0 = \left(\frac{\mathrm{i}}{8\pi} \right)^{1/2},$$

(6.52)
$$A_{10}|_0 = \left(\frac{\mathrm{i}}{8\pi} \right)^{1/2} \frac{9\beta_2(\partial_2 M)(x_{1,2}')}{16(\beta_1^2 + \beta_3^2)^2}.$$

The next order equation, for A_{01}, yields (cf. (6.20))

(6.53)
$$\widetilde{L}A_{01} + 4I_1 A_{01} + \widetilde{L}_1 A_{00} = 0,$$

where

$$\widetilde{L}_1 A = (\partial_{y_1} I_1)(\partial_{y_1} A) - (\partial_{\eta_2} I_1)(\partial_2^2 M)(x_1' + y_1, x_2' - Q)(\partial_{\eta_2} A)$$

$$+ \tfrac{1}{2}\left((\partial_{y_1}^2 I_1) - \frac{2P(\partial_{y_1} I_1)}{I_0} + \frac{P^2 I_1}{I_0^2} + 3I_2 - \frac{2I_1^2}{I_0} \right) A$$

$$+ \left[\left(\frac{QI_1}{I_0^2} - \frac{(\partial_{\eta_2} I_1)}{I_0} \right)(\partial_2 M)(x_1' + y_1, x_2' - Q) \right.$$

$$+ \left(\frac{Q(\partial_{\eta_2} I_1)}{I_0} - (\partial_{\eta_2}^2 I_1) \right)(\partial_2^2 M)(x_1' + y_1, x_2' - Q)$$

(6.54)
$$\left. + (\partial_{\eta_2} I_1)(\partial_{\eta_2} Q)(\partial_2^3 M)(x_1' + y_1, x_2' - Q) \right] A.$$

This equation simplifies from a computational point of view upon scaling $A_{01} = \mu^{-2}\bar{A}_{01}$; then

(6.55)
$$\widetilde{L}\bar{A}_{01} + \mu^2 \widetilde{L}_1 A_{00} = 0$$

with the initial condition $\bar{A}_{01}|_0 = 0$.

    We remark that the inhomogeneous term in (6.55) is continuous at the "source" $(x_{1,2}')$ and

$$A_{01} \to -\frac{3\alpha_2(\partial_2 M)(x_{1,2}')}{8\alpha_1^4 \mu} A_{00}|_0 \quad \text{as} \quad \mu \to 0,$$

TABLE 6.1
Relevant equations.

I_0	I_1	I_2	A_{00}	A_{10}	A_{01}
(6.33)	(6.37)	(6.39)	(6.47)	(6.48)	(6.55)

cf. (6.41) for Q and (6.42) for I_0. The inhomogeneous term in (6.48) is continuous at the "source" $(x'_{1,2})$ since

$$-\frac{3}{4}\frac{Q}{I_0^2}(\partial_2 M)(x'_1 + y_1, x'_2 - Q) \to \frac{3}{4}\frac{\alpha_2(\partial_2 M)(x'_{1,2})}{\alpha_1^4 \mu} \quad \text{as} \quad \mu \to 0.$$

Effective index of refraction and effective metric. Again, we introduce an *effective* index of refraction and *effective* horizontal distance as

$$(6.56) \qquad\qquad \nu \equiv [I_0 I_1]^{1/2},$$

$$(6.57) \qquad\qquad \chi_1 \equiv \left[\frac{I_0}{I_1}\right]^{1/2},$$

where the arguments are evaluated along the characteristics, whereas

$$(6.58) \qquad\qquad r = [\chi_1^2 + z^2]^{1/2}.$$

Then

$$(6.59) \quad G(x_{1,2}, x_3 - x'_3; x'_{1,2})$$
$$= \frac{k_0}{2\pi}\int_{\mathbb{R}}\frac{1}{(k_0\nu r)^{1/2}}\exp[\mathrm{i}k_0(\nu r + \eta_2 y_2)]\left\{A_{00} + \frac{1}{\mathrm{i}k_0}\left(\frac{A_{00}}{8\nu r} + A_{10}\right) + \ldots\right\}\mathrm{d}\eta_2,$$

cf. (6.1), (6.2), (6.19), which represents the outer solution. The equations to be evaluated or solved are listed in Table 6.1.

7. Uniform asymptotic expansions of the vertical slowness operator and the one-way wave propagator.

The square-root operator kernel. Using (4.8), upon carrying out repeated differentiation, we arrive at the uniform asymptotic expansion of the square-root operator kernel,

$$(7.1) \quad \mathcal{C}(x_{1,2}, x'_{1,2}; x'_3) = \frac{\mathrm{i}}{2\pi\omega\chi_1^3}\exp(\mathrm{i}k_0\nu\chi_1)\exp(\psi_{10})$$

$$\left\{(\mathrm{i}k_0\nu\chi_1 + 2\psi_{10})\left(1 + \frac{1}{\mathrm{i}k_0\nu\chi_1}\left(\frac{3}{8}\left(1 - \frac{\nu^3 I_2}{\chi_1}\right) + \nu(\nu\psi_{11} + \chi_1\psi_{20})\right)\right)\right.$$

$$\left. -\frac{1}{2}\left(1 + \frac{\nu^3 I_2}{\chi_1}\right) + \ldots\right\},$$

in the absence of caustics, and the outer expansion,

$$(7.2) \quad \mathcal{C}(x_{1,2}, x'_{1,2}; x'_3) = \frac{\mathrm{i}k_0}{\omega\pi}\int_{\mathbb{R}}\exp[\mathrm{i}k_0(\nu\chi_1 + \eta_2 y_2)]$$

$$\left\{\frac{1}{(\nu\chi_1)^{1/2}\chi_1^2}\left(\left(\mathrm{i}k_0\nu\chi_1 - \frac{3}{8}\right)A_{00} + 2\chi_1^2 A_{01} + \nu\chi_1 A_{10} + \ldots\right)\right\}\mathrm{d}\eta_2,$$

in the presence of a caustic, to the order considered.

The propagator kernel. Using (4.2), we arrive at the uniform asymptotic expansion of the one-way propagator kernel,

$$(7.3) \quad \mathcal{G}(x_{1,2}, x_3 - x_3'; x_{1,2}')$$

$$= -2(x_3 - x_3') \left\{ \left(\frac{\mathrm{i}k_0\nu}{r} + \frac{2\psi_{10}}{\chi_1^2} \right) G(x_{1,2}, x_3 - x_3'; x_{1,2}') \right.$$

$$- \frac{1}{4\pi r^3} \exp(\mathrm{i}k_0\nu r) \exp\left(\frac{\psi_{10} r^2}{\chi_1^2} \right)$$

$$\left. \left[-1 + \frac{1}{8\chi_1^2 r^2} \left(1 - \frac{\nu^3 I_2}{\chi_1} \right) (4\chi_1^4 + (x_3 - x_3')^2 (r^2 + \chi_1^2)) \right] + \dots \right\}$$

in the absence of caustics, and the outer expansion,

$$(7.4) \quad \mathcal{G}(x_{1,2}, x_3 - x_3'; x_{1,2}') = -\frac{k_0}{\pi} \int_{\mathbb{R}} \frac{(x_3 - x_3')}{r^2 (k_0 \nu r)^{1/2}}$$

$$\exp[\mathrm{i}k_0(\nu r + \eta_2 y_2)] \left\{ \left(\mathrm{i}k_0 \nu r - \frac{3}{8} \right) A_{00} + \nu r A_{10} + 2r^2 A_{01} + \dots \right\} \mathrm{d}\eta_2,$$

in the presence of a caustic, to the order considered. In both cases, we observe that

$$\lim_{x_3 \downarrow x_3'} \mathcal{G}(x_{1,2}, x_3 - x_3'; x_{1,2}') = \delta(x_{1,2} - x_{1,2}'),$$

as it should.

8. Discussion. One of the main objectives of directional wave field decomposition is the introduction of the concept of "tracing waves." A general theory for this, employing the complete generalized Bremmer coupling series, has been developed before. The application of the series, however, depends on solving an operator composition equation, the characteristic equation, and an associated one-way wave equation. In this paper, in smoothly varying media, we have obtained uniform asymptotic expansions for both solutions valid in the "high- and mid-frequency" wave regime.

The method of uniform asymptotics consists of three components: (i) the construction of a "far-field" or "outer" solution, representing an operator kernel away from its diagonal and obtained by microlocal techniques suppressing locally medium variations in the principal (here vertical) direction; (ii) the construction of a "near-field" or "inner" solution, representing the operator kernel near its diagonal and obtained mostly by Taylor-like expansions; (iii) matching the inner and outer solutions in a boundary layer to all orders considered.

The result is a one-way wave field representation that is truly more general than its microlocal counterpart. For example, the microlocal treatment of the one-way operator solutions to the characteristic equation would require cut-offs removing critical-angle scattering phenomena. Also, modal behavior is naturally included in our framework of uniform asymptotics. Conceptually, our theory is an intermediate between asymptotic-ray and full-wave theories in the sense that our theory is still asymptotic but valid in a much larger frequency band (see also Thomson [12]).

From a computational perspective, the uniform asymptotic one-way wave propagator falls into the category of propagators associated with the paraxial wave equation, the phase-screen or split-step Fourier approximation, the phase-shift-plus-interpolation

method, and so on. However, it does not suffer from any of the limitations of these approaches. A desirable feature of a closed-form solution as presented in this paper is its ease of use, in particular with a view to taking caustics into account. (In this context, for a comparison with a ray tracing approach, see Ziomek [13]).

Hidden in the uniform asymptotic expansions are certain aspects of homogenization: we have introduced an effective index of refraction and an effective metric, which follow from the actual medium variations and are evaluated by means of ray methods.

Throughout the paper, the configuration has been assumed to be three-dimensional. Previous two-dimensional results, obtained by more restrictive arguments, are recovered by assuming that $\partial_2 n \equiv 0$ and integrating the characteristic Green's function over y_2.

As a final remark, we indicate how variable density can be incorporated in the analysis. For details on how it affects the decomposition procedure, see de Hoop [1]. The key in the approach presented in this paper is the introduction of an effective wave speed, $c'^{-2}(x_{1,2},\omega) = c_0^{-2} n'^2(x_{1,2},\omega)$, with

$$n'^2(x_{1,2},\omega) = n^2(x_{1,2}) + c_0^2 \left[\frac{3[(D_1\rho)^2 + (D_2\rho)^2]}{4\rho^2} - \frac{(D_1^2 + D_2^2)\rho}{2\rho} \right], \quad \rho = \rho(x_1, x_2).$$

This change requires some straightforward adjustments of the asymptotic matching.

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