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Inverting the local geodesic ray transform of higher rank tensors

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Abstract

Consider a Riemannian manifold in dimension $n \geqslant 3$ with a strictly convex boundary. We prove the local invertibility, up to potential fields, of the geodesic ray transform on tensor fields of rank four near a boundary point. This problem is closely related to elastic qP-wave tomography. Under the condition that the manifold can be foliated with a continuous family of strictly convex hypersurfaces, the local invertibility implies a global result. One can straightforwardedly adapt the proof to show similar results for tensor fields of arbitrary rank.

Keywords: tensor tomography, elastic-wave travel-time tomography, scattering calculus

1. Introduction

We let $M \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary ∂M and $x = (x^1, x^2, x^3)$ be the Cartesian coordinates. The system of equations describing elastic waves reads

$$\rho \partial_t^2 u = \operatorname{div}(\mathbf{C}\varepsilon(u)). \tag{1.1}$$

Here, u denotes the displacement vector and

$$\varepsilon(u) = (\nabla u + (\nabla u)^T)/2 = (\varepsilon_{ij}(u)) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)$$

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the linear strain tensor which is the symmetric part of ∇u. Furthermore, $\mathbf{C} = (C_{ijkl}) = (C_{ijkl}(x))$ is the stiffness tensor and $\rho = \rho(x)$ is the density of mass.

The stiffness tensor is assumed to have the symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}.$$

The operator $\operatorname{div}(\mathbf{C}\varepsilon(\cdot))$ is elliptic if we additionally assume that there exists a $\delta > 0$ such that for any 3×3 real-valued symmetric matrix (ε_{ij}),

$$\sum_{i,j,k,l=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqslant \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2.$$

If the stiffness tensor $\mathbf{C}$ is isotropic, we have

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.2}$$

where λ, μ are referred to as the Lamé parameters. For isotropic elasticity there are two different wave-speeds, namely, P-wave (longitudinal wave) speed $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ and S-wave (transverse wave) speed $c_S = \sqrt{\mu/\rho}$. Then we can consider M as a manifold with metric $c_P^{-2}\mathrm{d}s^2$ or $c_S^{-2}\mathrm{d}s^2$. Correspondingly, we can view P waves traveling along geodesics in the Riemannian manifold $(M, c_P^{-2}\mathrm{d}s^2)$, and S waves traveling along geodesics in the Riemannian manifold $(M, c_S^{-2}\mathrm{d}s^2)$.

If there is an anisotropic perturbation a_{ijkl} around isotropy, that is,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + a_{ijkl},$$

the perturbation in the travel time of P-waves along a geodesic γ gives the following quantity [2]:

$$\int_\gamma \frac{a_{ijkl}}{\rho c_P^6}\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k\dot{\gamma}^l\mathrm{d}t. \tag{1.3}$$

Here, γ is a geodesic in $(M, c_P^{-2}\mathrm{d}s^2)$. The same quantity has been derived by a different perturbation analysis [13]. Equation (1.3) represents a geodesic ray transform of a four-tensor $b_{ijkl} = \frac{a_{ijkl}}{\rho c_P^6}$ in $(M, c_P^{-2}\mathrm{d}s^2)$.

Let (M, g) be a compact Riemannian manifold with boundary ∂M. The geodesic ray transform of a symmetric tensor field f of order m is given by

$$I_m f(\gamma) = \int_\gamma \langle f(\gamma(t)), \dot{\gamma}^m(t)\rangle\mathrm{d}t, \tag{1.4}$$

where, in local coordinates, $\langle f, v^m(t)\rangle = f_{i_1,\cdots,i_m}v^{i_1}\cdots v^{i_m}$, and γ runs over all geodesics with endpoints on ∂M. We note, here, that the tensor b in (1.3) is not fully symmetric. Thus, we introduce f as the symmetrization of b, and study the geodesic x-ray transform $I_4 f$. A general tensor with symmetry (1.2) has 21 unknowns, while a symmetric four-tensor has 15 unknowns. Therefore, we have already lost six components of $\mathbf{C}$ in the formulation of the problem.

It is known that potential tensor fields, i.e. $f = \mathrm{d}^s v$ with v a symmetric field of order $m - 1$ vanishing on ∂M ($m \geqslant 1$), are in the kernel of I_m. Here, d^s is the symmetric part of the covariant derivative ∇, which will be explicitly defined in (2.6). We say that I_m is s-injective if $I_m f = 0$ implies $f = \mathrm{d}^s v$ with $v|_{\partial M} = 0$. The s-injectivity of I_m has been extensively investigated, and we refer to [5, 17] for detailed reviews.

Assuming that M is simple, when ∂M is strictly convex and any two points in M are connected by a unique minimizing geodesic smoothly depending on the endpoints, it has been proved that I_0 is injective [7, 8], and I_1 is s-injective [1]. In dimension two, the s-injectivity of I_m for arbitrary m is proved in [9]. In dimension three or higher, the s-injectivity of $I_m, m \geqslant 2$ is still open. When (M, g) has negative sectional curvature [12], or is under certain other curvature conditions [3, 11, 13], the s-injectivity has been established. Without any curvature condition, it has been proved that the problem is Fredholm [15] (modulo potential fields) with a finite-dimensional smooth kernel. For analytic simple metrics, the uniqueness is proved using microlocal analytic continuation. With the Fredholm property, the uniqueness can be extended to an open and dense set of simple metrics in $C^k, k \gg 1$ containing analytic simple metrics.

In [18], Uhlmann and Vasy proved that, if ∂M is strictly convex at $p \in \partial M$ in dimension three or higher, $I_0 f(\gamma)$, for all geodesics localized in some suitable Ω near p, determines f near p. Furthermore, under some global convex 'foliation condition', it gives a global result via layer stripping techniques. Then, Stefanov *et al* gave corresponding results for I_1 and I_2 [16]. The key point is to show the ellipticity (under a suitable gauge condition) of a different version of the normal operator $I_m^* I_m$ as a scattering pseudodifferential operator. The calculation for I_1, I_2, which is already massive, is not observed to have an easy extension to I_m, $m \geqslant 3$. In this paper, we will prove the parallel results for I_4 occur for two main reasons: (1) the calculation arises naturally from elastic qP-wave tomography; (2) the scheme of the calculation needs to be general enough so that one can easily adapt the procedure to prove similar results for I_m with arbitrary m.

For an open set $O \subset M$, $O \cap \partial M \neq \emptyset$, we call γ an O-local geodesic if γ is a geodesic contained in O with endpoints in ∂M. We denote the set of O-local geodesics by $\mathcal{M}_O$. Note that $\mathcal{M}_O$ is an open subset of the set of all geodesics $\mathcal{M}$. The introduction of $\mathcal{M}$ and $\mathcal{M}_O$ can be found in [18]. We define the local geodesic ray transform of f as the collection $(I_m f)(\gamma)$ along all geodesics $\gamma \in \mathcal{M}_O$, that is, as the restriction of the geodesic ray transform to $\mathcal{M}_O$. We restrict ourselves to problem (1.4) with $m = 4$ from now on.

First, we consider M as a strictly convex domain in a Riemannian manifold $(\tilde{M}, g)$ (without boundary), with a boundary defining function ρ, such that $\rho \geqslant 0$ on M. As in [16, 18], we first study the invertibility of I_4 in a neighborhood of a point $p \in \partial M$ of the form $\{\tilde{x} > -c\}$, $c > 0$. Here, $\tilde{x}$ is a function with $\tilde{x}(p) = 0$, $\mathrm{d}\tilde{x}(p) = -\mathrm{d}\rho(p)$. We denote $\Omega = \Omega_c = \{x \geqslant 0, \rho \geqslant 0\}$, $x = x_c = \tilde{x} + c$. Using the local geodesic ray transform with Ω-local geodesics, we have the local injectivity result:

Theorem 1.1. *With $\Omega = \Omega_c$ as above, there is $c_0 > 0$ such that for $c \in (0, c_0)$ if $f \in L^2(\Omega)$ is a symmetric four-tensor. Then $f = u + \mathrm{d}^s v$, where $v \in \dot{H}^1_{\mathrm{loc}}(\Omega \setminus \{x = 0\})$, while $u \in L^2_{\mathrm{loc}}(\Omega \setminus \{x = 0\})$ can be stably determined from $I_4 f$ restricted to Ω-local geodesics in the following sense. There is a continuous map $I_4 f \mapsto u$, where for $s \geqslant 0$, $f \in H^s(\Omega)$, the H^{s-1} norm of u restricted to any compact subset of $\Omega \setminus \{x = 0\}$ is controlled by the H^s norm of $I_4 f$ restricted to the set of Ω-local geodesics.*

Replacing $\Omega_c = \{\tilde{x} > -c\} \cap M$ by $\Omega_{\tau,c} = \{\tau > \tilde{x} > -c + \tau\} \cap M$, c can be taken to be uniform in τ for τ in a compact set on which the strict concavity assumption on level sets of $\tilde{x}$ holds.

The Sobolev spaces $\dot{H}^1_{\mathrm{loc}}$ will be defined in section 3. As in [16, 18], the above theorem can be applied to obtain the following global result. Assume that $\tilde{x}$ is a globally defined function with level sets $\Sigma_t = \{\tilde{x} = t\}$ strictly concave (viewed from $\tilde{x}^{-1}(0, t)$) for $t \in (-T, 0]$, with $\tilde{x} \leqslant 0$ on the manifold M with the boundary. Assume, furthermore, that $\Sigma_0 = \partial M$ and $M \setminus \cup_{t \in (-T,0]} \Sigma_t$ has measure 0 or has an empty interior. We say that such an M satisfies the foliation condition.

Theorem 1.2. *Suppose M is compact. The geodesic ray transform is injective and has stable modulo potentials on the restriction of symmetric four-tensors f to $\tilde{x}^{-1}((-T,0])$ in the following sense. For all $\tau > -T$ there is $v \in \dot{H}^1_{\mathrm{loc}}(\tilde{x}^{-1}((\tau,0]))$ such that $f - \mathrm{d}^s v \in L^2_{\mathrm{loc}}(\tilde{x}^{-1}((\tau,0]))$ can be stably recovered from $I_4 f$. Here, for stability, we assume that $s \geqslant 0$, f is in an H^s-space, the norm on $I_4 f$ is an H^s-norm, while the norm for v is an H^{s-1}-norm.*

The foliation condition can be satisfied even in the presence of caustics. A Riemannian manifold $(M, c^{-2}(|x|)\mathrm{d}s^2)$ satisfying the Herglotz [4] and Wiechert and Zoeppritz [19] condition $\frac{\mathrm{d}}{\mathrm{d}r}\frac{r}{c(r)} > 0$ satisfies the foliation condition. The Euclidean spheres $|x| = r$ form a strictly convex foliation. In PREM (the preliminary reference earth model), this condition is essentially satisfied, but it might well be violated in other terrestrial planets. We note, here, that the condition does not exclude the existence of conjugate points. Further discussions on the foliation condition can be found in [10] and the references therein.

2. Pseudodifferential property

In Ω, we can use local coordinates (x, y), with x introduced above. We are interested in geodesics 'almost tangent' to level sets of $\tilde{x}$. Let $\gamma_{x,y,\lambda,\omega}$ be a geodesic in $\tilde{M}$ such that

$$\gamma_{x,y,\lambda,\omega}(0) = (x, y),\ \dot{\gamma}_{x,y,\lambda,\omega}(0) = (\lambda, \omega),$$

with $(x, y, \lambda, \omega) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-2}$. In order for $x \geqslant 0$ and λ to be sufficiently small, we need the geodesic $\gamma_{x,y,\lambda,\omega}(t)$ to stay in $x \geqslant 0$ as long as it is in M. Thus for $x = 0$, λ can only be 0. This is guaranteed if $|\lambda| < C_1\sqrt{x}$, for sufficiently small C_1. For convenience, we use a smaller range $|\lambda| \leqslant C_2 x$. We take χ to be a smooth, even, non-negative function with compact support (to be specified).

We denote

$$(I_4 f)(x, y, \lambda, \omega) = \int_{\mathbb{R}} \langle f(\gamma_{x,y,\lambda,\omega}(t)), \dot{\gamma}^4_{x,y,\lambda,\omega}(t)\rangle \mathrm{d}t. \tag{2.1}$$

We note here that we are only interested in f supported in $\overline{M}$, hence the above integration is actually along the segment of $\gamma_{x,y,\lambda,\omega}$ in M. On $u(x, y, \lambda, \omega)$, we define

$$\begin{aligned}(L_4 u)(x, y) = x^6 \int \chi(\lambda/x) u(x, y, \lambda, \omega) g_{\mathrm{sc}}(\lambda\partial_x + \omega\partial_y) \otimes g_{\mathrm{sc}}(\lambda\partial_x + \omega\partial_y)\\ \otimes g_{\mathrm{sc}}(\lambda\partial_x + \omega\partial_y) \otimes g_{\mathrm{sc}}(\lambda\partial_x + \omega\partial_y)\mathrm{d}\lambda\mathrm{d}\omega.\end{aligned} \tag{2.2}$$

We carry out the calculation on $X = \{x \geqslant 0\}$. Here, u is a (locally defined in the support of χ) function on the space of geodesics parametrized by (x, y, λ, ω), and g_{sc} maps vectors to covectors; g_{sc} is the scattering metric of the form

$$g_{\mathrm{sc}} = x^{-4}\mathrm{d}x^2 + x^{-2}h, \tag{2.3}$$

where $h(x, y)$ is a standard two-cotensor on X.

As in [16], we will show that $L_4 I_4$, conjugated by an exponential weight, is in Melrose's scattering pseudodifferential algebra (see [6] for an introduction). The ellipticity of the scattering pseudodifferential operator will be the main subject of this section. In local coordinates $(x, y^1, \cdots, y^{n-1})$, the scattering tangent bundle ${}^{\mathrm{sc}}TX$ has a local basis $x^2\partial_x, x\partial_{y^1}, \cdots, x\partial_{y^{n-1}}$, and the dual bundle ${}^{\mathrm{sc}}T^*X$ correspondingly has a local basis $\frac{\mathrm{d}x}{x^2}, \frac{\mathrm{d}y^1}{x}, \cdots, \frac{\mathrm{d}y^{n-1}}{x}$. We adopt the notation $\Psi^{m,l}_{\mathrm{sc}}(X)$ for the scattering pseudodifferential algebra introduced in [16]. We also use the notation ${}^{\mathrm{sc}}TX$, ${}^{\mathrm{sc}}T^*X$ and $\mathrm{Sym}^{m\,\mathrm{sc}}T^*X$ defined there in the following analogue of [16, proposition 3.1]

Proposition 2.1. *On symmetric four-tensors, the operator* $N_{\mathsf{F}} = \mathrm{e}^{-\mathsf{F}/x} L_4 I_4 \mathrm{e}^{\mathsf{F}/x}$ *lies in*

$$\Psi_{\mathrm{sc}}^{-1,0}(X; \mathrm{Sym}^{4\mathrm{sc}}T^*X, \mathrm{Sym}^{4\mathrm{sc}}T^*X),$$

for $\mathsf{F} > 0$.

We will, in fact, be working with $\mathrm{e}^{-\mathsf{F}/x} f$ from now on.

Proof. We employ the map introduced in [18],

$$\Gamma_+ : S\tilde{M} \times [0,\infty) \to [\tilde{M} \times \tilde{M}; \mathrm{diag}],\ \Gamma_+(x,y,\lambda,\omega,t) = (x,y,\gamma_{x,y,\lambda,\omega}(t)),$$

and similarly Γ_- while replacing $[0,\infty)$ by $(-\infty,0]$. We note, here, that $\Gamma_\pm$ are local diffeomorphisms [18, section 3.3]. We take the local coordinates from the two factors of $\tilde{M}$ as (x,y,x',y'). In the above, $[\tilde{M} \times \tilde{M}; \mathrm{diag}]$ is the *blow-up* of $\tilde{M}$ at the diagonal $(x,y) = (x',y')$, which essentially involves introducing polar coordinates around the diagonal. We write

$$(\gamma_{x,y,\lambda,\omega}(t), \dot{\gamma}_{x,y,\lambda,\omega}(t)) = (\mathsf{X}_{x,y,\lambda,\omega}(t), \mathsf{Y}_{x,y,\lambda,\omega}(t), \Lambda^\flat_{x,y,\lambda,\omega}(t), \Omega^\flat_{x,y,\lambda,\omega}(t)),$$

in coordinates (x,y,λ,ω) for the lifted geodesic $\gamma_{x,y,\lambda,\omega}(t)$. We use the coordinates on Melrose's scattering double space near the lifted scattering diagonal,

$$x,\ y,\ X = \frac{x'-x}{x^2},\ Y = \frac{y'-y}{x}$$

as in [18]. Then we use the coordinates on $[\tilde{M} \times \tilde{M}; \mathrm{diag}]$ in the region of interest $(|x-x'| < C|y-y'|)$

$$x,\ y,\ |y-y'|,\ \frac{x'-x}{|y-y'|},\ \frac{y'-y}{|y-y'|}.$$

Note that these are $x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}$, where $\hat{Y} = \frac{Y}{|Y|}$. Here, the norms are Euclidean norms. In these coordinates, we have the representation

$$\Gamma_+(x,y,\lambda,\omega,t) = \left(x,y,|y'-y|,\frac{x'-x}{|y'-y|},\frac{y'-y}{|y'-y|}\right),$$

$$\Gamma_-(x,y,\lambda,\omega,t) = \left(x,y,|y'-y|,-\frac{x'-x}{|y'-y|},-\frac{y'-y}{|y'-y|}\right).$$

As in [16], λ, ω are given in terms of x, x', y, y' as

$$(\Lambda \circ \Gamma_\pm^{-1})\left(x,y,x|Y|,\frac{xX}{|Y|},\hat{Y}\right) = x\frac{X - \alpha(x,y,x|Y|,\frac{xX}{|Y|},\hat{Y})|Y|^2}{|Y|} + x^2\tilde{\Lambda}_\pm\left(x,y,x|Y|,\frac{xX}{|Y|},\hat{Y}\right),$$

and

$$(\Omega \circ \Gamma_\pm^{-1})\left(x,y,x|Y|,\frac{xX}{|Y|},\hat{Y}\right) = \hat{Y} + x|Y|\tilde{\Omega}_\pm\left(x,y,x|Y|,\frac{xX}{|Y|},\hat{Y}\right),$$

where $\tilde{\Lambda}_\pm$ and $\tilde{\Omega}_\pm$ are smooth. Evaluating $(\Lambda^\flat_{x,y,\lambda,\omega}(t), \Omega^\flat_{x,y,\lambda,\omega}(t))$ at (x', y') gives us the tangent vector $\lambda'\partial_{x'} + \omega'\partial_{y'} = \dot{\gamma}_{x,y,\lambda,\omega}(t)$, where λ' is given in terms of x, x', y, y' by

$$\Lambda' \circ \Gamma_\pm^{-1} = x\frac{X + \alpha(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y})|Y|^2}{|Y|} + x^2|Y|^2\tilde{\Lambda}'\left(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}\right),$$

and ω' is given by

$$\Omega' \circ \Gamma_\pm^{-1} = \hat{Y} + x\tilde{\Omega}'\left(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}\right).$$

Here, $\tilde{\Gamma}', \tilde{\Omega}'$ are also smooth.

Then we can express $g_{\mathrm{sc}}(\lambda\partial_x + \omega\partial_y)$ and $\dot{\gamma}_{x,y,\lambda,\omega}(t)$ in terms of x, x', y, y' as in the proof of [16, proposition 3.1]. Finally, we obtain the Schwartz kernel of N_{F} on symmetric four-tensors:

$$\begin{aligned} K^\flat(x, y, X, Y) = \sum_\pm \mathrm{e}^{-\mathsf{F}X/(1+xX)}\chi\left(\frac{X - \alpha(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y})|Y|^2}{|Y|} + x\tilde{\Lambda}_\pm\left(x, y, x|Y|, \frac{x|X|}{|Y|}, \hat{Y}\right)\right) \\ \left[x^{-1}(\Lambda \circ \Gamma_\pm^{-1})\frac{\mathrm{d}x}{x^2} + (\Omega \circ \Gamma_\pm^{-1})\frac{h(\partial_y)}{x}\right]^4 \left[x^{-1}(\Lambda' \circ \Gamma_\pm^{-1})x^2\partial_x + (\Omega' \circ \Gamma_\pm^{-1})x\partial_y\right]^4 \\ |Y|^{-n+1}J_\pm\left(x, y, \frac{X}{|Y|}, |Y|, \hat{Y}\right). \end{aligned} \tag{2.4}$$

The density factor $J_\pm$ is smooth and positive, and $J_\pm = 1$ at $x = 0$. Due to the order x vanishing of Λ (and Λ'), the smoothness properties as a bundle endomorphism are similar to [16, proposition 3.1]. This proves the proposition. □

We denote T^mM to be the space of m-tensors and S^mM the space of symmetric m-tensors. Then the connection

$$\nabla : T^mM \to T^{m+1}M$$

is defined component-wisely as

$$\begin{aligned} \nabla_k u_{j_1,\cdots,j_m} &= u_{j_1,\cdots,j_m;k} \\ &= \frac{\partial}{\partial x^k}u_{j_1,\cdots,j_m} - \sum_{p=1}^m \Gamma^q_{k,j_p}u_{j_1,\cdots,j_{p-1},q,j_{p+1},\cdots,j_m}, \end{aligned} \tag{2.5}$$

where Γ is the Christoffel symbol with respect to the metric g. For $u \in T^mM$, we define its symmetrization as

$$\begin{aligned} \mathscr{S} : T^mM &\to S^mM \\ u &\mapsto w, \end{aligned}$$

with

$$w(v_1, \cdots, v_m) = \frac{1}{m!}\sum_\sigma u(v_{\sigma(1)}, \cdots, v_{\sigma(m)}),$$

where σ runs over all permutation groups of $(1, \cdots, m)$, and $v_j \in C^\infty(TM)$, $j = 1, \cdots, m$.

The symmetric differential $\mathrm{d}^s \in S^m M \to S^{m+1}M$ is defined as

$$\mathrm{d}^s = \mathscr{S}\nabla. \tag{2.6}$$

Here, d^s is different from the exterior differential d defined on the bundle of k-forms $\Lambda^k M$. (For its properties, see [13].) We also define $\mathrm{d}^s_{\mathsf{F}} = \mathrm{e}^{-\mathsf{F}/x}\mathrm{d}^s\mathrm{e}^{\mathsf{F}/x}$ and denote its adjoint with respect to the scattering metric g_{sc} (not g) as δ^s_{F}.

Now we turn our attention to $\mathrm{Sym}^{m\,\mathrm{sc}}T^*X$. For convenience of calculation, we will use the basis

$$\begin{gathered}
\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}
\end{gathered}$$

for three-tensors, and the basis

$$\begin{gathered}
\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}x}{x^2},\\
\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}x}{x^2}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x},\\
\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x}\otimes\frac{\mathrm{d}y}{x},
\end{gathered}$$

for four-tensors. Thus for $\mathrm{Sym}^{3\,\mathrm{sc}}T^*X$, we use the basis

$$\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2},\ 2\times\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x},\ 2\times\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}$$

and for $\mathrm{Sym}^{4\,\mathrm{sc}}T^*X$, we use the basis

$$\begin{aligned}
&\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2},\ 4\times\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x},\ 6\times\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x},\\
&\qquad 4\times\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}.
\end{aligned}$$

In the above, $\otimes_s$ denotes the symmetric product, for example, $a\otimes_s b = \mathscr{S}(a\otimes b)$. With this convention, a symmetric four-tensor, f, that can be represented under the above basis as

$$f = \begin{pmatrix} f_{xxxx}\\ f_{xxxy}\\ f_{xxyy}\\ f_{xyyy}\\ f_{yyyy}\end{pmatrix}$$

is the tensor

$$
\begin{aligned}
f =& f_{xxxx}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}+4\times f_{xxxy^i}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\\
&+6\times f_{xxy^iy^j}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}+4\times f_{xy^iy^jy^k}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\otimes_s\frac{\mathrm{d}y^k}{x}\\
&+f_{y^iy^jy^ky^l}\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\otimes_s\frac{\mathrm{d}y^k}{x}\otimes_s\frac{\mathrm{d}y^l}{x}.
\end{aligned}
$$

Lemma 2.2. *On symmetric four-tensors,* $\mathrm{d}^s_{\mathsf{F}}\delta^s_{\mathsf{F}}\in\mathrm{Diff}^{2,0}_{\mathrm{sc}}(X;\mathrm{Sym}^{4\mathrm{sc}}T^*X,\mathrm{Sym}^{4\mathrm{sc}}T^*X)$ *has the principal symbol*

$$
\begin{aligned}
\mathfrak{D}(x,y,\xi,\eta)&=\begin{pmatrix}\xi+\mathrm{i}\mathsf{F} & 0 & 0 & 0\\ \frac{1}{4}\eta\otimes & \frac{3}{4}(\xi+\mathrm{i}\mathsf{F}) & 0 & 0\\ a^\flat & \frac{1}{2}\eta\otimes_s & \frac{1}{2}(\xi+\mathrm{i}\mathsf{F}) & 0\\ 0 & b^\flat & \frac{3}{4}\eta\otimes_s & \frac{1}{4}(\xi+\mathrm{i}\mathsf{F})\\ 0 & 0 & c^\flat & \eta\otimes_s\end{pmatrix}\begin{pmatrix}\xi-\mathrm{i}\mathsf{F} & \iota_\eta & 6\langle a^\flat,\cdot\rangle & 0 & 0\\ 0 & (\xi-\mathrm{i}\mathsf{F}) & \iota^s_\eta & \frac{4}{3}\langle b^\flat,\cdot\rangle & 0\\ 0 & 0 & (\xi-\mathrm{i}\mathsf{F}) & \iota^s_\eta & \frac{1}{3}\langle c^\flat,\cdot\rangle\\ 0 & 0 & 0 & (\xi-\mathrm{i}\mathsf{F}) & \iota^s_\eta\end{pmatrix}\\
&=\begin{pmatrix}|\xi|^2+\mathsf{F}^2 & (\xi+\mathrm{i}\mathsf{F})\iota_\eta & 6(\xi+\mathrm{i}\mathsf{F})\langle a^\flat,\cdot\rangle & 0 & 0\\ \frac{1}{4}(\xi-\mathrm{i}\mathsf{F})\eta\otimes & \frac{1}{4}(\eta\otimes)\iota_\eta+\frac{1}{4}(|\xi|^2+\mathsf{F}^2) & \mathfrak{D}_{23} & \mathfrak{D}_{24} & 0\\ (\xi-\mathrm{i}\mathsf{F})a^\flat & a^\flat\iota_\eta+\frac{1}{2}(\xi-\mathrm{i}\mathsf{F})\eta\otimes_s & \mathfrak{D}_{33} & \mathfrak{D}_{34} & \mathfrak{D}_{35}\\ 0 & (\xi-\mathrm{i}\mathsf{F})b^\flat & \mathfrak{D}_{43} & \mathfrak{D}_{44} & \mathfrak{D}_{45}\\ 0 & 0 & \mathfrak{D}_{53} & \mathfrak{D}_{54} & \mathfrak{D}_{55}\end{pmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
&\mathfrak{D}_{23}=\frac{3}{2}\eta\otimes\langle a^\flat,\cdot\rangle+\frac{3}{4}(\xi+\mathrm{i}\mathsf{F})\iota^s_\eta,\qquad \mathfrak{D}_{24}=(\xi+\mathrm{i}\mathsf{F})\otimes\langle b^\flat,\cdot\rangle,\\
&\mathfrak{D}_{33}=6a^\flat\langle a^\flat,\cdot\rangle+\frac{1}{2}(\eta\otimes)\iota_\eta+\frac{1}{2}(|\xi|^2+\mathsf{F}^2),\qquad \mathfrak{D}_{34}=\frac{2}{3}\eta\otimes\langle b^\flat,\cdot\rangle+\frac{1}{2}(\xi+\mathrm{i}\mathsf{F})\otimes\iota^s_\eta,\\
&\mathfrak{D}_{35}=\frac{1}{6}(\xi+\mathrm{i}\mathsf{F})\otimes\langle c^\flat,\cdot\rangle,\qquad \mathfrak{D}_{43}=b^\flat\iota^s_\eta+\frac{3}{4}(\xi-\mathrm{i}\mathsf{F})\eta\otimes_s,\\
&\mathfrak{D}_{44}=\frac{4}{3}b^\flat\langle b^\flat,\cdot\rangle+\frac{3}{4}(\eta\otimes)\iota^s_\eta+\frac{1}{4}(|\xi|^2+\mathsf{F}^2),\qquad \mathfrak{D}_{45}=\frac{1}{4}\eta\otimes\langle c^\flat,\cdot\rangle+\frac{1}{4}(\xi+\mathrm{i}\mathsf{F})\otimes\iota^s_\eta,\\
&\mathfrak{D}_{53}=(\xi-\mathrm{i}\mathsf{F})c^\flat,\qquad \mathfrak{D}_{54}=c^\flat\iota^s_\eta+(\xi-\mathrm{i}\mathsf{F})\eta\otimes_s,\qquad \mathfrak{D}_{55}=\frac{1}{3}c^\flat\langle c^\flat,\cdot\rangle+\eta\otimes_s\iota_\eta.
\end{aligned}
$$

The terms $a^\flat, b^\flat, c^\flat$ *are defined below in (2.7).*

Proof. We can assume g_{sc} to be of the form $x^{-4}\mathrm{d}x^2+x^{-2}\mathrm{d}y^2$ to perform the calculation, where $\mathrm{d}y^2$ is the flat metric. See the discussion in the proof of lemma 3.2 in [16] for more details. Assume that a symmetric three-tensor f can be written as

$$
\begin{aligned}
f =& f_{xxx}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}+3\times f_{xxy^i}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\\
&+3\times f_{xy^iy^j}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}+f_{y^iy^jy^k}\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\otimes_s\frac{\mathrm{d}y^k}{x}.
\end{aligned}
$$

By calculation

$$\begin{aligned}
(\nabla f)_{xxxx} &= x^{-6}\partial_x f_{xxx} + O(x^{-7}),\\
(\nabla f)_{xxxy^i} &= x^{-6}\partial_{y^i} f_{xxx} + O(x^{-6}),\\
(\nabla f)_{xxy^ix} &= x^{-5}\partial_x f_{xxy^i} + O(x^{-6}),\\
(\nabla f)_{xxy^iy^j} &= x^{-5}\partial_{y^j} f_{xxy^i} + x^{-6}a_1(f_{xxx}) + O(x^{-5}),\\
(\nabla f)_{xy^iy^jx} &= x^{-4}\partial_x f_{xy^iy^j} + O(x^{-5}),\\
(\nabla f)_{xy^iy^jy^k} &= x^{-4}\partial_{y^k} f_{xy^iy^j} + x^{-5}b_1(f_{xxy}) + O(x^{-4}),\\
(\nabla f)_{y^iy^jy^kx} &= x^{-3}\partial_x f_{y^iy^jy^k} + O(x^{-4}),\\
(\nabla f)_{y^iy^jy^ky^l} &= x^{-3}\partial_{y^l} f_{y^iy^jy^k} + x^{-4}c_1(f_{xyy}) + O(x^{-3}).
\end{aligned}$$

Here a_1, b_1, c_1 come from the contributions of the Christoffel symbol Γ in equation (2.5). Then, we derive

$$\begin{aligned}
\mathrm{d}^s f =& x^2\partial_x f_{xxx}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\\
&+4\times\left(\frac{1}{4}x\partial_{y^i}f_{xxx}+\frac{3}{4}x^2\partial_x f_{xxy^i}\right)\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\\
&+6\times\left(\frac{1}{2}\mathrm{Sym}_y(x\partial_{y^j}f_{xxy^i})+\frac{1}{2}x^2\partial_x f_{xy^iy^j}+a^\flat(f_{xxx})\right)\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\\
&+4\times\left(\frac{3}{4}\mathrm{Sym}_y(x\partial_{y^k}f_{xy^iy^j})+\frac{1}{4}x^2\partial_x f_{y^iy^jy^k}+b^\flat(f_{xxy})\right)\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\otimes_s\frac{\mathrm{d}y^k}{x}\\
&+\left(\mathrm{Sym}_y(x\partial_{y^l}f_{y^iy^jy^k})+c^\flat(f_{xyy})\right)\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\otimes_s\frac{\mathrm{d}y^k}{x}\otimes_s\frac{\mathrm{d}y^l}{x}+\text{l.o.t.}.
\end{aligned} \tag{2.7}$$

In the above, Sym_y is defined as

$$\mathrm{Sym}_y(v_{y^{k_1},\cdots,y^{k_m}}) = \frac{1}{m!}\sum_\sigma v_{y^{k_{\sigma(1)}},\cdots,y^{k_{\sigma(m)}}}.$$

It follows that d^s has the principal symbol

$$\begin{pmatrix}
\xi & 0 & 0 & 0\\
\frac{1}{4}\eta\otimes & \frac{3}{4}\xi & 0 & 0\\
a^\flat & \frac{1}{2}\eta\otimes_s & \frac{1}{2}\xi & 0\\
0 & b^\flat & \frac{3}{4}\eta\otimes_s & \frac{1}{4}\xi\\
0 & 0 & c^\flat & \eta\otimes_s
\end{pmatrix}.$$

The term $\eta\otimes_s$ in the (32)-block has (iji')-entry (corresponding to the (ij) entry of the symmetric two-tensor on Y and the i' entry of the one-tensor)

$$\frac{1}{2}(\eta_i\delta_{ji'}+\eta_j\delta_{ii'}).$$

The term $\eta\otimes_s$ in the (43)-block has $(ijki'j')$-entry (corresponding to the (ijk) entry of the symmetric three-tensor and the $i'j'$ entry of the three-tensor)

$$\frac{1}{6}(\eta_i\delta_{ji'}\delta_{kj'}+\eta_i\delta_{ki'}\delta_{jj'}+\eta_j\delta_{ii'}\delta_{kj'}+\eta_j\delta_{ki'}\delta_{jj'}+\eta_k\delta_{ii'}\delta_{jj'}+\eta_k\delta_{ji'}\delta_{ij'}).$$

The term $\eta\otimes_s$ in the (54)-block has $(ijkli'j'k')$-entry (corresponding to the $(ijkl)$ entry of the symmetric 4-tensor and the $i'j'k'$ entry of the three-tensor)

$$\frac{1}{24}\Big(\sum_\sigma \eta_i\delta_{j\tau(\sigma(1))}\delta_{k\tau(\sigma(2))}\delta_{l\tau(\sigma(3))}+\sum_\sigma \eta_j\delta_{i\tau(\sigma(1))}\delta_{k\tau(\sigma(2))}\delta_{l\tau(\sigma(3))}$$
$$+\sum_\sigma \eta_k\delta_{i\tau(\sigma(1))}\delta_{j\tau(\sigma(2))}\delta_{l\tau(\sigma(3))}+\sum_\sigma \eta_l\delta_{i\tau(\sigma(1))}\delta_{j\tau(\sigma(2))}\delta_{k\tau(\sigma(3))}\Big).$$

Here, σ runs over all permutations of (123), and $\tau(1)=i', \tau(2)=j', \tau(3)=k'$.

We note that $a^\flat$ maps a 0-tensor (smooth function) to a symmetric two-tensor, $b^\flat$ maps a symmetric one-tensor to a symmetric three-tensor, and $c^\flat$ maps a symmetric two-tensor to a symmetric four-tensor. They are symmetrizations of a, b, c respectively. We note here that they only play a role in the principal symbol at the boundary. Then the symbol of $\mathrm{d}^s_{\mathsf{F}}=\mathrm{e}^{-\mathsf{F}/x}\mathrm{d}^s\mathrm{e}^{\mathsf{F}/x}$ is given by

$$\begin{pmatrix} \xi+\mathrm{i}\mathsf{F} & 0 & 0 & 0\\ \frac{1}{4}\eta\otimes & \frac{3}{4}(\xi+\mathrm{i}\mathsf{F}) & 0 & 0\\ a^\flat & \frac{1}{2}\eta\otimes_s & \frac{1}{2}(\xi+\mathrm{i}\mathsf{F}) & 0\\ 0 & b^\flat & \frac{3}{4}\eta\otimes_s & \frac{1}{4}(\xi+\mathrm{i}\mathsf{F})\\ 0 & 0 & c^\flat & \eta\otimes_s \end{pmatrix}.$$

With our basis for symmetric tensors, the inner product with respect to g_{sc} on four-tensors is given by the (block-)matrix

$$M(4)=\begin{pmatrix} 1 &&&&\\ & 4\times\mathrm{Id} &&&\\ && 6\times\mathrm{Id} &&\\ &&& 4\times\mathrm{Id} &\\ &&&& \mathrm{Id}\end{pmatrix} \tag{2.8}$$

and for three-tensors

$$M(3)=\begin{pmatrix} 1 &&&\\ & 3\times\mathrm{Id} &&\\ && 3\times\mathrm{Id} &\\ &&& \mathrm{Id}\end{pmatrix}. \tag{2.9}$$

If A maps a symmetric m_1-tensor to a symmetric m_2-tensor, we call B the (m_2, m_1)-adjoint of A if

$$\langle Bf_2, f_1\rangle_{M(m_1)}=\langle f_2, Af_1\rangle_{M(m_2)}.$$

It is easy to check that

$$B=M(m_1)^{-1}A^*M(m_2).$$

If $m_1=m_2=m$, we say that A is (m,m)-self-adjoint if $B=A$.

It follows that δ^s_{F} has a symbol given by the $(3,4)$-adjoint of that of $\mathrm{d}^s_{\mathsf{F}}$,

$$\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta & 6\langle a^\flat, \cdot\rangle & 0 & 0 \\ 0 & (\xi - \mathrm{i}\mathsf{F}) & \iota^s_\eta & \frac{4}{3}\langle b^\flat, \cdot\rangle & 0 \\ 0 & 0 & (\xi - \mathrm{i}\mathsf{F}) & \iota^s_\eta & \frac{1}{3}\langle c^\flat, \cdot\rangle \\ 0 & 0 & 0 & (\xi - \mathrm{i}\mathsf{F}) & \iota^s_\eta \end{pmatrix}.$$

Remaining tedious calculations complete the proof. □

Next, we show that N_{F} is elliptic as a scattering pseudodifferential operator, for which we need to show that the principal symbol is positive definite at both the *fiber infinity* of ${}^{sc}T^*X$ (when $|\zeta|$ is sufficiently large, where $\zeta = (\xi, \eta)$ is the Fourier dual variable of (X, Y)) and *finite points* of ${}^{sc}T^*X$ (when x is sufficiently close to 0 and $|\zeta|$ is relatively small to x^{-1}). By smoothness of the Schwartz kernel, we only need to investigate the principal symbol at $x = 0$. See the discussion before lemma 4.1 in [10] for more details.

Lemma 2.3. *On symmetric four-tensors, N_{F} is elliptic at fiber infinity in ${}^{sc}T^*X$ when restricted to the kernel of the principal symbol of δ^s_{F}.*

Proof. With the notation,

$$S = \frac{X - \alpha(\hat{Y})|Y|^2}{|Y|}, \ \hat{Y} = \frac{Y}{|Y|},$$

by (2.4), the Schwartz kernel of N_{F} at the scattering front face $x = 0$ is given by

$$\mathrm{e}^{-\mathsf{F}X}|Y|^{-n+1}\chi(S)\left[S\frac{\mathrm{d}x}{x^2} + \hat{Y}\cdot\frac{\mathrm{d}y}{x}\right]^4\left[(S + 2\alpha|Y|)(x^2\partial_x) + \hat{Y}\cdot(x\partial_y)\right]^4. \quad (2.10)$$

On a symmetric four-tensor of the form

$$\begin{aligned} f =& f_{xxxx}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2} + 4f_{xxxy}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x} \\ &+ 6f_{xxyy}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x} + 4f_{xyyy}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x} \quad (2.11) \\ &+ f_{yyyy}\cdot\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}, \end{aligned}$$

we have

$$\begin{aligned} &\left[(S + 2\alpha|Y|)(x^2\partial_x) + \hat{Y}\cdot(x\partial_y)\right]^4 f \\ =& (S + 2\alpha|Y|)^4 f_{xxxx} + 4(S + 2\alpha|Y|)^3\langle\hat{Y}, f_{xxxy}\rangle + 6(S + 2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle \\ &+ 4(S + 2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle + \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle. \end{aligned}$$

On a scalar a,

$$\left[S\frac{\mathrm{d}x}{x^2}+\hat{Y}\cdot\frac{\mathrm{d}y}{x}\right]^4 a = aS^4\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}+4aS^3\hat{Y}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}$$
$$+6aS^2\hat{Y}\otimes\hat{Y}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}+4aS\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}$$
$$+a\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\cdot\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}.$$

Thus, under the basis of symmetric four-tensors, we write

$$\left[S\frac{\mathrm{d}x}{x^2}+\hat{Y}\cdot\frac{\mathrm{d}y}{x}\right]^4\left[(S+2\alpha|Y|)(x^2\partial_x)+\hat{Y}\cdot(x\partial_y)\right]^4$$
$$=\begin{pmatrix} S^4\\ S^3\hat{Y}\\ S^2\hat{Y}\otimes\hat{Y}\\ S\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(S+2\alpha|Y|)^4\\ 4(S+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(S+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(S+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T.$$

One can easily verify that the above matrix is $(4,4)$-self-adjoint. We note that there are no coefficients $(1,4,6,4,1)$ for the column vector. In coordinates on the support of χ,

$$x, y, |Y|, \frac{X}{|Y|}, \hat{Y},$$

we can rewrite the kernel as

$$\mathrm{e}^{-\mathsf{F}X}|Y|^{-n+1}\chi(S)\begin{pmatrix} S^4\\ S^3\hat{Y}\\ S^2\hat{Y}\otimes\hat{Y}\\ S\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(S+2\alpha|Y|)^4\\ 4(S+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(S+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(S+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T.$$

We use the change of coordinates $(X,Y)\to(|Y|,\tilde{S},\hat{Y})$. To analyze the principal symbol at fiber infinity, we need to evaluate the (X,Y)-Fourier transform as $|\zeta|\to+\infty$. As discussed in the proof of [16, lemma 3.4], the leading order behavior of the Fourier transform as $|\zeta|\to+\infty$ can be obtained by integrating the restriction of the Schwartz kernel to the front face $|Y|=0$, dropping the singular factor $|Y|^{-n+1}$, along the equatorial sphere:

$$\tilde{S}\xi+\hat{Y}\cdot\eta=0. \tag{2.12}$$

Following the discussion around (3.8) in [18], we need to integrate

$$\chi(\tilde{S})\begin{pmatrix} \tilde{S}^4\\ \tilde{S}^3\hat{Y}\\ \tilde{S}^2\hat{Y}\otimes\hat{Y}\\ \tilde{S}\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}\tilde{S}^4\\ 4\tilde{S}^3\langle\hat{Y},\cdot\rangle\\ 6\tilde{S}^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4\tilde{S}\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \tag{2.13}$$

on this sphere.

For a symmetric four-tensor of the form (2.11) in the kernel of the principal symbol of δ^s_{F}, we have by lemma 2.2 that

$$\begin{aligned}
&\xi f_{xxxx} + \langle \eta, f_{xxxy}\rangle = 0,\\
&\xi f_{xxxy} + \langle \eta, f_{xxyy}\rangle = 0,\\
&\xi f_{xxyy} + \langle \eta, f_{xyyy}\rangle = 0,\\
&\xi f_{xyyy} + \langle \eta, f_{yyyy}\rangle = 0.
\end{aligned} \tag{2.14}$$

Moreover, f is in the kernel of (2.13) if and only if

$$\begin{aligned}
\tilde{S}^4 f_{xxxx} + 4\tilde{S}^3\langle \hat{Y}, f_{xxxy}\rangle + 6\tilde{S}^2\langle \hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle\\
+4\tilde{S}\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle + \langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0.
\end{aligned} \tag{2.15}$$

Suppose a symmetric four-tensor f satisfies (2.14) and (2.15) for $(\tilde{S}, \hat{Y})$ such that (2.12) holds. We will consider two cases, $\xi = 0$ and $\xi \neq 0$.

Case 1 : $\xi \neq 0$. If $\eta = 0$, we have directly from (2.14) that

$$f_{xxxx}, f_{xxxy}, f_{xxyy}, f_{xyyy}$$

all vanish. Then from (2.15), we have

$$\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0.$$

Therefore, $f_{yyyy} = 0$, since $\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}$ spans the space of all symmetric four-tensors with $\eta = 0$. If $\eta \neq 0$, we calculate successively,

$$\begin{aligned}
&f_{xyyy} = -\langle \frac{\eta}{\xi}, f_{yyyy}\rangle,\\
&\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle = -\langle \frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxyy} = -\langle \frac{\eta}{\xi}, f_{xyyy}\rangle = \langle \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}, f_{yyyy}\rangle,\\
&\langle \hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle = \langle \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxxy} = -\langle \frac{\eta}{\xi}, f_{xxyy}\rangle = -\langle \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}, f_{yyyy}\rangle,\\
&\langle \hat{Y}, f_{xxxy}\rangle = -\langle \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxxx} = -\langle \frac{\eta}{\xi}, f_{xxxy}\rangle = \langle \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}, f_{yyyy}\rangle.
\end{aligned}$$

With $\tilde{S} = -\frac{\hat{Y}\cdot\eta}{\xi}$, (2.15) gives

$$\begin{aligned}
\Big\langle \left(\frac{\hat{Y}\cdot\eta}{\xi}\right)^4 \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi} + 4\left(\frac{\hat{Y}\cdot\eta}{\xi}\right)^3 \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y} + 6\left(\frac{\hat{Y}\cdot\eta}{\xi}\right)^2 \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}\\
+4\left(\frac{\hat{Y}\cdot\eta}{\xi}\right)\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} + \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\Big\rangle = 0.
\end{aligned} \tag{2.16}$$

Now we take $\hat{Y} = \epsilon\hat{\eta} + (1-\epsilon^2)^{1/2}\hat{Y}^\perp$, where $\hat{Y}^\perp$ is a unit vector orthogonal to $\hat{\eta}$. Substituting $\hat{Y}$ into (2.16), we find that

$$\begin{aligned}\Big\langle \epsilon^4 & \left(\frac{|\eta|^8}{\xi^8} + \frac{4|\eta|^6}{\xi^6} + \frac{6|\eta|^4}{\xi^4} + \frac{4|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta} \\ &+ 4\epsilon^3(1-\epsilon^2)^{1/2}\left(\frac{|\eta|^6}{\xi^6} + \frac{3|\eta|^4}{\xi^4} + \frac{3|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\otimes\hat{Y}^\perp \\ &+ 6\epsilon^2(1-\epsilon^2)\left(\frac{|\eta|^4}{\xi^4} + \frac{2|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{\eta}\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp \\ &+ 4\epsilon(1-\epsilon^2)^{3/2}\left(\frac{|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp \\ &+ (1-\epsilon^2)^2\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp, f_{yyyy}\Big\rangle = 0.\end{aligned} \tag{2.17}$$

Taking $\epsilon = 0$ in (2.17), we have

$$\langle \hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp, f_{yyyy}\rangle = 0.$$

Since $\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp$ spans $\eta^\perp\otimes\eta^\perp\otimes\eta^\perp\otimes\eta^\perp$, we conclude that f_{yyyy} is orthogonal to every element of $\eta^\perp\otimes\eta^\perp\otimes\eta^\perp\otimes\eta^\perp$. Taking 1st, 2nd, 3rd and 4th order derivatives of (2.17) at $\epsilon = 0$, it follows that f_{yyyy} is orthogonal to

$$\hat{\eta}\otimes\hat{\eta}^\perp\otimes\eta^\perp\otimes\hat{\eta}^\perp, \qquad \hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}^\perp\otimes\hat{\eta}^\perp,$$
$$\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}^\perp, \qquad \hat{\eta}\otimes\eta\otimes\hat{\eta}\otimes\hat{\eta},$$

respectively. We then finally conclude that f_{yyyy} vanishes, and then the whole tensor f vanishes by (2.14).

Case 2 : $\xi = 0$ (and so $\eta \neq 0$). Now (2.12) is equivalent to $\eta\cdot\hat{Y} = 0$, and (2.14) reduces to

$$\begin{aligned}&\langle\hat{\eta}, f_{xxxy}\rangle = 0,\\ &\langle\hat{\eta}, f_{xxyy}\rangle = 0,\\ &\langle\hat{\eta}, f_{xyyy}\rangle = 0,\\ &\langle\hat{\eta}, f_{yyyy}\rangle = 0.\end{aligned} \tag{2.18}$$

We differentiate (2.15) with respect to $\tilde{S}$ up to four times, evaluated at $\tilde{S} = 0$, and find that

$$\begin{aligned} f_{xxxx} &= 0,\\ \langle\hat{Y}, f_{xxxy}\rangle &= 0,\\ \langle\hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle &= 0,\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle &= 0,\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle &= 0.\end{aligned} \tag{2.19}$$

Combining the identities in (2.18) and (2.19), we conclude that $f = 0$. □

Lemma 2.4. *There exists* $\mathsf{F}_0 > 0$ *such that on symmetric four-tensors* N_F *is elliptic at a finite set of points in* ${}^{\mathrm{sc}}T^*X$ *when restricted to the kernel of the principal symbol of* δ^s_F *for any* $\mathsf{F} > \mathsf{F}_0$.

Proof. Taking $\chi(s) = \mathrm{e}^{-s^2/(2\nu(\hat{Y}))}$, so $\hat{\chi}(\cdot) = c\sqrt{\nu}\mathrm{e}^{-\nu|\cdot|^2/2}$. We get the *X*-Fourier transform of the Schwartz kernel at the front face $x = 0$:

$$
\begin{aligned}
&\mathcal{F}_X K^\flat(0, y, |Y|, \frac{\xi}{|Y|}, \hat{Y})\\
=&|Y|^{2-n}\mathrm{e}^{-\mathrm{i}\alpha(-\xi-\mathrm{iF})|Y|^2} \begin{pmatrix} D_\sigma^4 \\ -D_\sigma^3 \hat{Y} \\ D_\sigma^2 \hat{Y}\otimes\hat{Y} \\ -D_\sigma \hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \end{pmatrix} \otimes \begin{pmatrix} (-D_\sigma + 2\alpha|Y|)^4 \\ 4(-D_\sigma + 2\alpha|Y|)^3\langle\hat{Y}, \cdot\rangle \\ 6(-D_\sigma + 2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ 4(-D_\sigma + 2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \end{pmatrix}^T \hat{\chi}((-\xi - \mathrm{iF})|Y|)\\
=&c\sqrt{\nu}|Y|^{2-n}\mathrm{e}^{\mathrm{i}\alpha(\xi+\mathrm{iF})|Y|^2} \begin{pmatrix} D_\sigma^4 \\ -D_\sigma^3 \hat{Y} \\ D_\sigma^2 \hat{Y}\otimes\hat{Y} \\ -D_\sigma \hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \end{pmatrix} \otimes \begin{pmatrix} (-D_\sigma + 2\alpha|Y|)^4 \\ 4(-D_\sigma + 2\alpha|Y|)^3\langle\hat{Y}, \cdot\rangle \\ 6(-D_\sigma + 2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ 4(-D_\sigma + 2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \end{pmatrix}^T \mathrm{e}^{-\nu(\xi+\mathrm{iF})^2|Y|^2/2}.
\end{aligned}
$$

Here D_σ denotes the differentiation of the argument of $\hat{\chi}$. Then we compute the *Y*-Fourier transform, which in polar coordinates takes the form,

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-2}}\int_0^\infty \mathrm{e}^{-\mathrm{i}|Y|\hat{Y}\cdot\eta}|Y|^{2-n}\mathrm{e}^{\mathrm{i}\alpha(\xi+\mathrm{iF})|Y|^2}\\
&\qquad \begin{pmatrix} D_\sigma^4 \\ -D_\sigma^3 \hat{Y} \\ D_\sigma^2 \hat{Y}\otimes\hat{Y} \\ -D_\sigma \hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \end{pmatrix} \otimes \begin{pmatrix} (-D_\sigma + 2\alpha|Y|)^4 \\ 4(-D_\sigma + 2\alpha|Y|)^3\langle\hat{Y}, \cdot\rangle \\ 6(-D_\sigma + 2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ 4(-D_\sigma + 2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \end{pmatrix}^T \mathrm{e}^{-\nu(\xi+\mathrm{iF})^2|Y|^2/2}|Y|^{n-2}\mathrm{d}|Y|\mathrm{d}\hat{Y}.
\end{aligned}
$$

We denote

$$\phi(\xi, \hat{Y}) = \nu(\hat{Y})(\xi + \mathrm{iF})^2 - 2\mathrm{i}\alpha(\xi + \mathrm{iF}).$$

By explicitly evaluating the derivatives, the above integral yields

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-2}}\int_0^\infty \mathrm{e}^{-\mathrm{i}|Y|\hat{Y}\cdot\eta} \begin{pmatrix} \mathrm{i}^4\nu^4(\xi+\mathrm{iF})^4|Y|^4 \\ \mathrm{i}^3\nu^3(\xi+\mathrm{iF})^3|Y|^3\hat{Y} \\ \mathrm{i}^2\nu^2(\xi+\mathrm{iF})^2|Y|^2\hat{Y}\otimes\hat{Y} \\ \mathrm{i}\nu(\xi+\mathrm{iF})|Y|\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \end{pmatrix} \otimes \begin{pmatrix} (\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^4|Y|^4 \\ 4(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^3|Y|^3\langle\hat{Y}, \cdot\rangle \\ 6(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^2|Y|^2\langle\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ 4(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)|Y|\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, \cdot\rangle \end{pmatrix}^T\\
&\qquad\times\mathrm{e}^{-\phi|Y|^2/2}\mathrm{d}|Y|\mathrm{d}\hat{Y}.
\end{aligned}
$$

We extend the integral in $|Y|$ to $\mathbb{R}$, replacing it by a variable *t*, and using the fact that the integrand is invariant under the joint change of variables $t \to -t$ and $\hat{Y} \to -\hat{Y}$. This gives

$$\int_{\mathbb{S}^{n-2}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}t\hat{Y}\cdot\eta}\begin{pmatrix}\mathrm{i}^4\nu^4(\xi+\mathrm{i}\mathsf{F})^4t^4\\ \mathrm{i}^3\nu^3(\xi+\mathrm{i}\mathsf{F})^3t^3\hat{Y}\\ \mathrm{i}^2\nu^2(\xi+\mathrm{i}\mathsf{F})^2t^2\hat{Y}\otimes\hat{Y}\\ \mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})t\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^4t^4\\ 4(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^3t^3\langle\hat{Y},\cdot\rangle\\ 6(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^2t^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)t\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \times\mathrm{e}^{-\phi t^2/2}\mathrm{d}t\mathrm{d}\hat{Y}.$$

Now the t integral is a Fourier transform evaluated at $-\hat{Y}\cdot\eta$, under which multiplication by t becomes $D_{\hat{Y}\cdot\eta}$. We also note that the Fourier transform of $\mathrm{e}^{-\phi(\xi,\hat{Y})t^2/2}$ is a constant multiple of

$$\phi(\xi,\hat{Y})^{-1/2}\mathrm{e}^{-(\hat{Y}\cdot\eta)^2/(2\phi(\xi,\hat{Y}))}. \tag{2.20}$$

Thus we are left with

$$\int_{\mathbb{S}^{n-2}}\phi(\xi,\hat{Y})^{-1/2}\begin{pmatrix}\mathrm{i}^4\nu^4(\xi+\mathrm{i}\mathsf{F})^4D^4_{\hat{Y}\cdot\eta}\\ \mathrm{i}^3\nu^3(\xi+\mathrm{i}\mathsf{F})^3D^3_{\hat{Y}\cdot\eta}\hat{Y}\\ \mathrm{i}^2\nu^2(\xi+\mathrm{i}\mathsf{F})^2D^2_{\hat{Y}\cdot\eta}\hat{Y}\otimes\hat{Y}\\ \mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})D_{\hat{Y}\cdot\eta}\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^4D^4_{\hat{Y}\cdot\eta}\\ 4(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^3\langle\hat{Y},\cdot\rangle D^3_{\hat{Y}\cdot\eta}\\ 6(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle D^2_{\hat{Y}\cdot\eta}\\ 4(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle D_{\hat{Y}\cdot\eta}\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \times\mathrm{e}^{-(\hat{Y}\cdot\eta)^2/(2\phi(\xi,\hat{Y}))}\mathrm{d}\hat{Y}.$$

We now take a semiclassical point of view setting $h=\mathsf{F}^{-1}$ as a small parameter and rescaling

$$\xi_{\mathsf{F}}=\mathsf{F}^{-1}\xi,\ \eta_{\mathsf{F}}=\mathsf{F}^{-1}\eta.$$

We let $\nu=\mathsf{F}^{-1}\alpha$, with

$$\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha=\nu(\xi-\mathrm{i}\mathsf{F})$$

and

$$\phi=(\xi+\mathrm{i}\mathsf{F})(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)=\nu(\xi^2+\mathsf{F}^2).$$

Then

$$\hat{Y}\cdot\eta=h^{-1}\hat{Y}\cdot\eta_{\mathsf{F}},\ D_{\hat{Y}\cdot\eta}=hD_{\hat{Y}\cdot\eta_{\mathsf{F}}},\ \phi(\xi,\hat{Y})=\frac{\alpha(\xi_{\mathsf{F}}^2+1)}{h},\ \nu(\xi+\mathrm{i}\mathsf{F})=(\xi_{\mathsf{F}}+\mathrm{i})\alpha.$$

As $h\to 0$, leading order terms of the above operator give

$$\int_{\mathbb{S}^{n-2}}h^{1/2}\alpha^{-1/2}(\xi_{\mathsf{F}}^2+1)^{-1/2}\begin{pmatrix}(\xi_{\mathsf{F}}+\mathrm{i})^4\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^4\\ -(\xi_{\mathsf{F}}+\mathrm{i})^3\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^3\hat{Y}\\ (\xi_{\mathsf{F}}+\mathrm{i})^2\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^2\hat{Y}\otimes\hat{Y}\\ -(\xi_{\mathsf{F}}+\mathrm{i})\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(\xi_{\mathsf{F}}-\mathrm{i})^4\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^4\\ -4(\xi_{\mathsf{F}}-\mathrm{i})^3\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^3\langle\hat{Y},\cdot\rangle\\ 6(\xi_{\mathsf{F}}-\mathrm{i})^2\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ -4(\xi_{\mathsf{F}}-\mathrm{i})\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \times\mathrm{e}^{-(\hat{Y}\cdot\eta_{\mathsf{F}})^2/(2h(\xi_{\mathsf{F}}^2+1)\alpha)}\mathrm{d}\hat{Y},$$

which is a superposition of nonnegative operators. Hence, if a vector is in the kernel of the above operator, it must be in the kernel of

$$\begin{pmatrix} (\xi_{\mathsf{F}}+\mathrm{i})^4\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^4 \\ -(\xi_{\mathsf{F}}+\mathrm{i})^3\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^3\hat{Y} \\ (\xi_{\mathsf{F}}+\mathrm{i})^2\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^2\hat{Y}\otimes\hat{Y} \\ -(\xi_{\mathsf{F}}+\mathrm{i})\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \end{pmatrix} \otimes \begin{pmatrix} (\xi_{\mathsf{F}}-\mathrm{i})^4\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^4 \\ -4(\xi_{\mathsf{F}}-\mathrm{i})^3\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^3\langle\hat{Y},\cdot\rangle \\ 6(\xi_{\mathsf{F}}-\mathrm{i})^2\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle \\ -4(\xi_{\mathsf{F}}-\mathrm{i})\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \end{pmatrix}^T \tag{2.21}$$

for any $\hat{Y}$. We note, here, that (2.21) is a multiple of a projection and is $(4,4)$-self-adjoint. We then denote

$$\begin{aligned}
\mathfrak{C}_4 &= (\xi_{\mathsf{F}}-\mathrm{i})^4\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^4, \\
\mathfrak{C}_3 &= -(\xi_{\mathsf{F}}-\mathrm{i})^3\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^3, \\
\mathfrak{C}_2 &= (\xi_{\mathsf{F}}-\mathrm{i})^2\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^2, \\
\mathfrak{C}_1 &= -\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right).
\end{aligned}$$

For a symmetric 4-tensor of the form (2.11) in the kernel of the principal symbol of δ^s_{F} (also considered semiclassically), we have by lemma 2.2 that

$$\begin{aligned}
(\xi_{\mathsf{F}}-\mathrm{i})f_{xxxx}+\langle\eta_{\mathsf{F}},f_{xxxy}\rangle &= 0, \\
(\xi_{\mathsf{F}}-\mathrm{i})f_{xxxy}+\langle\eta_{\mathsf{F}},f_{xxyy}\rangle &= 0, \\
(\xi_{\mathsf{F}}-\mathrm{i})f_{xxyy}+\langle\eta_{\mathsf{F}},f_{xyyy}\rangle &= 0, \\
(\xi_{\mathsf{F}}-\mathrm{i})f_{xyyy}+\langle\eta_{\mathsf{F}},f_{yyyy}\rangle &= 0,
\end{aligned} \tag{2.22}$$

if we let $h\to 0$. Moreover, f is in the kernel of (2.21) if and only if

$$\begin{pmatrix} (\xi_{\mathsf{F}}-\mathrm{i})^4\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^4 \\ -4(\xi_{\mathsf{F}}-\mathrm{i})^3\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^3\langle\hat{Y},\cdot\rangle \\ 6(\xi_{\mathsf{F}}-\mathrm{i})^2\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle \\ -4(\xi_{\mathsf{F}}-\mathrm{i})\left(\frac{\hat{Y}\cdot\eta_{\mathsf{F}}}{\xi_{\mathsf{F}}^2+1}\right)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \end{pmatrix}^T f = 0$$

or, equivalently,

$$\mathfrak{C}_4 f_{xxxx} + 4\mathfrak{C}_3\langle \hat{Y}, f_{xxxy}\rangle + 6\mathfrak{C}_2\langle \hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle + 4\mathfrak{C}_1\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle + \langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0. \tag{2.23}$$

Then we calculate, successively,

$$\begin{aligned}
&f_{xyyy} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{yyyy}\rangle,\\
&\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxyy} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{xyyy}\rangle = (\xi_{\mathsf{F}} - \mathrm{i})^{-2}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}, f_{yyyy}\rangle,\\
&\langle \hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle = (\xi_{\mathsf{F}} - \mathrm{i})^{-2}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxxy} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{xxyy}\rangle = -(\xi_{\mathsf{F}} - \mathrm{i})^{-3}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}, f_{yyyy}\rangle,\\
&\langle \hat{Y}, f_{xxxy}\rangle = -(\xi_{\mathsf{F}} - \mathrm{i})^{-3}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxxx} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{xxxy}\rangle = (\xi_{\mathsf{F}} - \mathrm{i})^{-4}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}, f_{yyyy}\rangle.
\end{aligned}$$

We observe that

$$\mathfrak{C}_j = (-1)^j(\xi_{\mathsf{F}}^2 + 1)^{-j}(\xi_{\mathsf{F}} - \mathrm{i})^j\rho^j \text{ with } \rho = \hat{Y}\cdot\eta_{\mathsf{F}}.$$

Hence, using (2.22),

$$\langle \otimes_{i=1}^4((\xi_{\mathsf{F}}^2+1)^{-1}\rho\eta_{\mathsf{F}} + \hat{Y}), f_{yyyy}\rangle = 0.$$

If $\eta = 0$, then

$$\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0.$$

Since $\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}$ spans the space of all symmetric four-tensors, we conclude that $f_{yyyy} = 0$ and, hence, $f = 0$.

If $\eta_{\mathsf{F}} \neq 0$, we take $\hat{Y} = \epsilon\hat{\eta}_{\mathsf{F}} + (1-\epsilon^2)^{1/2}\hat{Y}^\perp$, where $\hat{Y}^\perp$ is orthogonal to $\hat{\eta}_{\mathsf{F}}$. Then by (2.22), we have

$$\begin{aligned}
\Bigg\langle &\epsilon^4\left(1+\frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2+1}\right)^4 \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\\
&+ 4\epsilon^3(1-\epsilon^2)^{1/2}\left(1+\frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2+1}\right)^3 \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^\perp\\
&+ 6\epsilon^2(1-\epsilon^2)\left(1+\frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2+1}\right)^2 \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\\
&+ 4\epsilon(1-\epsilon^2)^{3/2}\left(1+\frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2+1}\right) \hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\\
&+ (1-\epsilon^2)^2\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp, f_{yyyy}\Bigg\rangle = 0.
\end{aligned} \tag{2.24}$$

Similar to the proof of lemma 2.3, we take derivatives of (2.24) up to order four at $\epsilon = 0$; it follows that f_{yyyy} is orthogonal to

$$\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp},\quad \hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp},\quad \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp},$$
$$\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp},\quad \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}.$$

Then, $f = 0$.

We conclude that for sufficiently large $\mathsf{F} > 0$, one has ellipticity at all finite points. □

With lemmas 2.3 and 2.4, we obtain the following proposition by similar arguments as in the proof of [16, proposition 3.3]

Proposition 2.5. *There exists $\mathsf{F}_0 > 0$ such that for $\mathsf{F} > \mathsf{F}_0$ the following holds. Given $\tilde{\Omega}$, a neighborhood of $X \cap M = \{x \geqslant 0, \rho > 0\}$ in X; for a suitable choice of the cutoff $\chi \in C_c^{\infty}(\mathbb{R})$ and of $M \in \Psi_{\mathrm{sc}}^{-3,0}(X; \mathrm{Sym}^{3\mathrm{sc}}T^*X, \mathrm{Sym}^{3\mathrm{sc}}T^*X)$, the operator*

$$A_{\mathsf{F}} = N_{\mathsf{F}} + \mathrm{d}_{\mathsf{F}}^s M \delta_{\mathsf{F}}^s,\quad N_{\mathsf{F}} = \mathrm{e}^{-\mathsf{F}/x} L I_4 \mathrm{e}^{\mathsf{F}/x},\ \mathrm{d}_{\mathsf{F}}^s = \mathrm{e}^{-\mathsf{F}/x}\mathrm{d}^s \mathrm{e}^{\mathsf{F}/x},$$

is elliptic in $\Psi_{\mathrm{sc}}^{-1,0}(X; \mathrm{Sym}^{4\mathrm{sc}}T^*X, \mathrm{Sym}^{4\mathrm{sc}}T^*X)$ in $\tilde{\Omega}$.

3. Proofs of the main results

We prove the injectivity of I_4 with the gauge condition $\delta_{\mathsf{F}}^s f_{\mathsf{F}} = 0$ in $\Omega = \Omega_c$, where $f_{\mathsf{F}} = \mathrm{e}^{-\mathsf{F}/x} f$. Based on the discussion in [16, section 4], we first need to check the invertibility of $\Delta_{\mathsf{F},s}$. Here, $\Delta_{\mathsf{F},s} = \delta_{\mathsf{F}}^s \mathrm{d}_{\mathsf{F}}^s$ is the 'solenoidal Witten Laplacian' which we will show as invertible with the desired boundary condition. Similar results for I_1 and I_2 are provided in section 4 of [16].

Lemma 3.1. *There exists $\mathsf{F}_0 > 0$ such that for $\mathsf{F} \geqslant \mathsf{F}_0$ the operator $\Delta_F^s = \delta_{\mathsf{F}}^s \mathrm{d}_{\mathsf{F}}^s$ is (joint) elliptic in $\mathrm{Diff}_{\mathrm{sc}}^{2,0}(X; \mathrm{Sym}^{3\mathrm{sc}}T^*X, \mathrm{Sym}^{3\mathrm{sc}}T^*X)$ on symmetric three-tensors. In fact, on symmetric three-tensors*

$$\delta_{\mathsf{F}}^s \mathrm{d}_{\mathsf{F}}^s = \frac{1}{4}\nabla_{\mathsf{F}}^* \nabla_{\mathsf{F}} + \frac{3}{4}\mathrm{d}_{\mathsf{F}}^s \delta_{\mathsf{F}}^s + A + R, \tag{3.1}$$

where $R \in x\mathrm{Diff}_{\mathrm{sc}}^1(X; \mathrm{Sym}^{3\mathrm{sc}}T^*X, \mathrm{Sym}^{3\mathrm{sc}}T^*X)$, $A \in \mathrm{Diff}_{\mathrm{sc}}^1(X; \mathrm{Sym}^{3\mathrm{sc}}T^*X, \mathrm{Sym}^{3\mathrm{sc}}T^*X)$ and $\nabla_{\mathsf{F}} = \mathrm{e}^{-\mathsf{F}/x}\nabla \mathrm{e}^{\mathsf{F}/x}$, with the ∇ gradient relative to g_{sc} (not g), $\mathrm{d}_{\mathsf{F}} = \mathrm{e}^{-\mathsf{F}/x}\mathrm{d}\mathrm{e}^{\mathsf{F}/x}$ the exterior derivative on symmetric three-tensors, while δ_{F} is its adjoint on symmetric three-tensors.

Proof. By calculation and lemma 2.2, Δ_{F}^s has the symbol

$$\begin{pmatrix} \xi^2 + \mathsf{F}^2 + \frac{1}{4}|\eta|^2 & \frac{3}{4}(\xi + \mathrm{i}\mathsf{F})\iota_\eta & 0 & 0 \\ \frac{1}{4}(\xi - \mathrm{i}\mathsf{F})\eta\otimes & \frac{3}{4}(\xi^2 + \mathsf{F}^2) + \frac{1}{2}\iota_\eta^s \eta\otimes_s & \frac{1}{2}(\xi + \mathrm{i}\mathsf{F})\iota_\eta^s & 0 \\ 0 & \frac{1}{2}(\xi - \mathrm{i}\mathsf{F})\eta\otimes_s & \frac{1}{2}(\xi^2 + \mathsf{F}^2) + \frac{3}{4}\iota_\eta^s \eta\otimes_s & \frac{1}{4}(\xi + \mathrm{i}\mathsf{F})\iota_\eta^s \\ 0 & 0 & \frac{3}{4}(\xi - \mathrm{i}\mathsf{F})\eta\otimes_s & \frac{1}{4}(\xi^2 + \mathsf{F}^2) + \iota_\eta^s \eta\otimes_s \end{pmatrix}$$
$$+ \begin{pmatrix} 6\langle a^\flat, \cdot\rangle a^\flat & 3\langle a, \cdot\rangle\eta\otimes_s & 3(\xi + \mathrm{i}\mathsf{F})\langle a^\flat, \cdot\rangle & 0 \\ \iota_\eta^s a & \frac{4}{3}\langle b^\flat, \cdot\rangle b^\flat & \langle b^\flat, \cdot\rangle\eta\otimes_s & \frac{1}{3}(\xi + \mathrm{i}\mathsf{F})\langle b^\flat, \cdot\rangle \\ (\xi - \mathrm{i}\mathsf{F})a^\flat & \iota_\eta^s b^\flat & \langle c^\flat, \cdot\rangle c^\flat & \frac{1}{3}\langle c^\flat, \cdot\rangle\eta\otimes_s \\ 0 & (\xi - \mathrm{i}\mathsf{F})b^\flat & \iota_\eta^s c^\flat & 0 \end{pmatrix}.$$

Here, $\iota^s_\eta \eta\otimes_s$ at $(2,2)$-block has the (i_1', i_2')-entry

$$\frac{1}{2}(|\eta|^2\delta_{i_1',i_2'} + \eta_{i_1'}\eta_{i_2'}).$$

$\iota^s_\eta \eta\otimes_s$ at $(3,3)$-block has (i_1', j_1', i_2', j_2')-entry

$$\frac{1}{6}(|\eta|^2\delta_{i_1',i_2'}\delta_{j_1',j_2'} + |\eta|^2\delta_{i_1',j_2'}\delta_{j_1',i_2'} + \eta_{i_1'}\eta_{i_2'}\delta_{j_1'j_2'} + \eta_{i_1'}\eta_{j_2'}\delta_{i_1'j_2'} + \eta_{i_2'}\eta_{j_1'}\delta_{i_1'j_2'} + \eta_{j_1'}\eta_{j_2'}\delta_{i_1'i_2'})$$

and $\iota^s_\eta \eta\otimes_s$ at $(4,4)$-block has $(i_1', j_1', k_1', i_2', j_2', k_2')$-entry

$$\begin{aligned}
\frac{1}{24}\Big(&|\eta|^2(\delta_{i_1'i_2'}\delta_{j_1'j_2'}\delta_{k_1'k_2'} + \delta_{i_1'j_2'}\delta_{j_1'i_2'}\delta_{k_1'k_2'} + \delta_{i_1'k_2'}\delta_{j_1'j_2'}\delta_{k_1'i_2'} \\
&\qquad + \delta_{i_1'i_2'}\delta_{j_1'k_2'}\delta_{k_1'j_2'} + \delta_{i_1'j_2'}\delta_{j_1'k_2'}\delta_{k_1'i_2'} + \delta_{i_1'k_2'}\delta_{j_1'i_2'}\delta_{k_1'j_2'}) \\
&+ \eta_{i_1'}\eta_{i_2'}(\delta_{j_1'j_2'}\delta_{k_1'k_2'} + \delta_{j_1'k_2'}\delta_{k_1'j_2'}) + \eta_{i_1'}\eta_{j_2'}(\delta_{j_1'i_2'}\delta_{k_1'k_2'} + \delta_{j_1'k_2'}\delta_{i_1'k_2'}) \\
&+ \eta_{i_1'}\eta_{k_2'}(\delta_{j_1'j_2'}\delta_{k_1'i_2'} + \delta_{j_1'i_2'}\delta_{k_1'j_2'}) + \eta_{j_1'}\eta_{k_2'}(\delta_{i_1'j_2'}\delta_{k_1'i_2'} + \delta_{i_1'i_2'}\delta_{k_1'j_2'}) \\
&+ \eta_{j_1'}\eta_{j_2'}(\delta_{i_1'i_2'}\delta_{k_1'k_2'} + \delta_{i_1'k_2'}\delta_{i_1'k_2'}) + \eta_{k_1'}\eta_{k_2'}(\delta_{i_1'i_2'}\delta_{j_1'j_2'} + \delta_{i_1'j_2'}\delta_{j_1'i_2'}) \\
&+ \eta_{j_1'}\eta_{i_2'}(\delta_{i_1'j_2'}\delta_{k_1'k_2'} + \delta_{i_1'k_2'}\delta_{j_1'k_2'}) + \eta_{k_1'}\eta_{i_2'}(\delta_{j_1'j_2'}\delta_{i_1'k_2'} + \delta_{j_1'k_2'}\delta_{i_1'j_2'}) \\
&+ \eta_{j_1'}\eta_{j_2'}(\delta_{i_1'k_2'}\delta_{j_1'i_2'} + \delta_{i_1'i_2'}\delta_{j_1'k_2'})\Big).
\end{aligned}$$

We note that the gradient ∇ maps a symmetric three-tensor to a (non-symmetric) four-tensor.

We introduce some further notation. We let A be a matrix of blocks, with

$$A_{\downarrow\times k}$$

representing

$$\left.\begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}\right\} k-\text{tuple}.$$

Also, we write

$$A_{\rightarrow\times k}$$

representing

$$\begin{pmatrix} A & \cdots & A \end{pmatrix}.$$

Then we use the basis for four-tensors (not the symmetric ones) and symmetric three-tensors, under which the principal symbol of ∇_{F} relative to g_{sc} (not g) is

$$\begin{pmatrix}
\begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix} & & & \\
& \begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix}_{\downarrow\times 3} & & \\
& & \begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix}_{\downarrow\times 3} & \\
& & & \begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix}
\end{pmatrix}.$$

The number of rows is 16. Thus ∇_{F}^* has the principal symbol,

$$\begin{pmatrix} \begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix} & & & \\ & \frac{1}{3}\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix}_{\to \times 3} & & \\ & & \frac{1}{3}\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix}_{\to \times 3} & \\ & & & \begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix} \end{pmatrix}.$$

Then $\nabla_{\mathsf{F}}^* \nabla_{\mathsf{F}}$ has the symbol

$$\begin{pmatrix} \xi^2 + \mathsf{F}^2 + |\eta|^2 & 0 & 0 & 0 \\ 0 & \xi^2 + \mathsf{F}^2 + |\eta|^2 & 0 & 0 \\ 0 & 0 & \xi^2 + \mathsf{F}^2 + |\eta|^2 & 0 \\ 0 & 0 & 0 & \xi^2 + \mathsf{F}^2 + |\eta|^2 \end{pmatrix}. \tag{3.2}$$

Similar to our calculation in the proof of lemma 2.2, we obtain the principal symbol of $\mathrm{d}_{\mathsf{F}}^s \delta_{\mathsf{F}}^s$ on symmetric three-tensors,

$$\begin{pmatrix} \xi^2 + \mathsf{F}^2 & (\xi + \mathrm{i}\mathsf{F})\iota_\eta & 0 & 0 \\ \frac{1}{3}(\xi - \mathrm{i}\mathsf{F})\eta\otimes & \frac{2}{3}(\xi^2 + \mathsf{F}^2) + \frac{1}{3}\eta \otimes \iota_\eta & \frac{2}{3}(\xi + \mathrm{i}\mathsf{F})\iota_\eta^s & 0 \\ 0 & \frac{2}{3}(\xi - \mathrm{i}\mathsf{F})\eta\otimes_s & \frac{1}{3}(\xi + \mathsf{F}^2) + \frac{2}{3}\eta \otimes_s \iota_\eta & \frac{1}{3}(\xi + \mathsf{F})\iota_\eta^s \\ 0 & 0 & (\xi - \mathrm{i}\mathsf{F})\eta\otimes_s & \eta \otimes_s \iota_\eta \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 3(\xi + \mathrm{i}\mathsf{F})\langle d^\flat, \cdot\rangle & 0 \\ 0 & 0 & \eta\langle d^\flat, \cdot\rangle & \frac{1}{3}(\xi + \mathrm{i}\mathsf{F})\langle e^\flat, \cdot\rangle \\ (\xi - \mathrm{i}\mathsf{F})d^\flat & d^\flat \iota_\eta & 3d^\flat\langle d^\flat, \cdot\rangle & \frac{1}{3}\eta \otimes_s \langle e^\flat, \cdot\rangle \\ 0 & (\xi + \mathrm{i}\mathsf{F})e^\flat & e^\flat \iota_\eta^s & \frac{1}{2}e^\flat\langle e^\flat, \cdot\rangle \end{pmatrix}.$$

Here, $\eta \otimes \iota_\eta$ at the $(2, 2)$-block has (i'_1, i'_2)-entry

$$\eta_{i'_1, i'_2},$$

$\eta \otimes_s \iota_\eta$ at the $(3, 3)$-block has (i'_1, j'_1, i'_2, j'_2)-entry

$$\frac{1}{4}(\eta_{i'_1}\eta_{i'_2}\delta_{j'_1 j'_2} + \eta_{i'_1}\eta_{j'_2}\delta_{i'_1 j'_2} + \eta_{i'_2}\eta_{j'_1}\delta_{i'_1 j'_2} + \eta_{j'_1}\eta_{j'_2}\delta_{i'_1 i'_2})$$

and $\eta \otimes_s \iota_\eta$ at the $(4, 4)$-block has $(i'_1, j'_1, k'_1, i'_2, j'_2, k'_2)$-entry

$$\begin{aligned} \frac{1}{18}\Big(& \eta_{i'_1}\eta_{i'_2}(\delta_{j'_1 j'_2}\delta_{k'_1 k'_2} + \delta_{j'_1 k'_2}\delta_{k'_1 j'_2}) + \eta_{i'_1}\eta_{j'_2}(\delta_{j'_1 i'_2}\delta_{k'_1 k'_2} + \delta_{j'_1 k'_2}\delta_{i'_1 k'_2}) + \eta_{i'_1}\eta_{k'_2}(\delta_{j'_1 j'_2}\delta_{k'_1 i'_2} + \delta_{j'_1 i'_2}\delta_{k'_1 j'_2}) \\ & + \eta_{j'_1}\eta_{k'_2}(\delta_{i'_1 j'_2}\delta_{k'_1 i'_2} + \delta_{i'_1 i'_2}\delta_{k'_1 j'_2}) + \eta_{j'_1}\eta_{j'_2}(\delta_{i'_1 i'_2}\delta_{k'_1 k'_2} + \delta_{i'_1 k'_2}\delta_{i'_1 k'_2}) + \eta_{k'_1}\eta_{k'_2}(\delta_{i'_1 i'_2}\delta_{j'_1 j'_2} + \delta_{i'_1 j'_2}\delta_{j'_1 i'_2}) \\ & + \eta_{j'_1}\eta_{i'_2}(\delta_{i'_1 j'_2}\delta_{k'_1 k'_2} + \delta_{i'_1 k'_2}\delta_{j'_1 k'_2}) + \eta_{k'_1}\eta_{i'_2}(\delta_{j'_1 j'_2}\delta_{i'_1 k'_2} + \delta_{j'_1 k'_2}\delta_{i'_1 j'_2}) + \eta_{j'_1}\eta_{j'_2}(\delta_{i'_1 k'_2}\delta_{j'_1 i'_2} + \delta_{i'_1 i'_2}\delta_{j'_1 k'_2})\Big). \end{aligned}$$

We note that the principal symbol of $\delta_{\mathsf{F}}^s \mathrm{d}_{\mathsf{F}}^s$ is the same as that of $\frac{1}{4}\nabla_{\mathsf{F}}^*\nabla_{\mathsf{F}} + \frac{3}{4}\mathrm{d}_{\mathsf{F}}^s\delta_{\mathsf{F}}^s$, which is positive definite with a lower bound $\frac{1}{4}(\xi^2 + \mathsf{F}^2 + |\eta|^2)$. Suppose $a^\flat, b^\flat, c^\flat, d^\flat, d^\flat$ have a common bound C, then A has a bound $C^2 + C|\eta| + C\mathsf{F} + C\xi \leqslant C'(1 + \epsilon^{-1}) + \epsilon(\xi^2 + \mathsf{F}^2 + |\eta|^2)$. Then we can choose $\mathsf{F} > 0$ large enough, and complete the proof. $\square$

We need to consider the operator properties in several larger domains Ω_j containing Ω. Let Ω_j be a domain in M with boundary $\partial\Omega_j$ transversal to ∂X. Let $\dot{H}^{m,l}_{\mathrm{sc}}(\Omega_j)$ be the subspace of $H^{m,l}_{\mathrm{sc}}(X)$ consisting of distributions supported in $\overline{\Omega_j}$, and let $\bar{H}^{m,l}_{\mathrm{sc}}(\Omega_j)$ be the space of restrictions of elements of $H^{m,l}_{\mathrm{sc}}(X)$ to Ω_j. Thus, $\dot{H}^{m,l}_{\mathrm{sc}}(\Omega_j)^* = \bar{H}^{-m,-l}_{\mathrm{sc}}(\Omega_j)$.

Lemma 3.2. *There exists $\mathsf{F}_0 > 0$ such that for $\mathsf{F} \geqslant \mathsf{F}_0$, the operator $\Delta_{\mathsf{F},s} = \delta^s_{\mathsf{F}} \mathrm{d}^s_{\mathsf{F}}$, considered as a map $\dot{H}^{1,0}_{\mathrm{sc}} \to (\dot{H}^{1,0}_{\mathrm{sc}})^* = \bar{H}^{-1,0}_{\mathrm{sc}}$, is invertible.*

Proof. Since δ^s_{F} is defined as the adjoint of $\mathrm{d}^s_{\mathsf{F}}$ relative to the scattering metric, we have

$$\begin{aligned}\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}} &= \langle \mathrm{d}^s_{\mathsf{F}} u, \mathrm{d}^s_{\mathsf{F}} u\rangle = \langle \Delta_{\mathsf{F},s} u, u\rangle \\ &\leqslant \|\Delta_{\mathsf{F},s} u\|_{\bar{H}^{-1,0}_{\mathrm{sc}}} \|u\|_{\dot{H}^{1,0}_{\mathrm{sc}}} \leqslant \epsilon^{-1} \|\Delta_{\mathsf{F},s} u\|^2_{\bar{H}^{-1,0}_{\mathrm{sc}}} + \epsilon \|u\|^2_{\dot{H}^{1,0}_{\mathrm{sc}}}.\end{aligned} \tag{3.3}$$

By (3.1) and (3.2), we have

$$\delta^s_{\mathsf{F}} \mathrm{d}^s_{\mathsf{F}} = \frac{1}{4}\nabla^*\nabla + \frac{1}{4}\mathsf{F}^2 + \frac{3}{4}\mathrm{d}^s_{\mathsf{F}} \delta^s_{\mathsf{F}} + A + \tilde{R}, \tag{3.4}$$

where $A \in \mathrm{Diff}^1_{\mathrm{sc}}(X)$ with

$$|\langle Au, u\rangle| \leqslant C\|u\|_{\dot{H}^{1,0}_{\mathrm{sc}}} \|u\|_{L^2_{\mathrm{sc}}} + C\mathsf{F}\|u\|^2_{L^2_{\mathrm{sc}}},$$

and $\tilde{R} \in x\mathrm{Diff}^1_{\mathrm{sc}}(X)$. This follows by rewriting $\nabla^*_{\mathsf{F}}\nabla_{\mathsf{F}}$ using (3.2), which modifies R in (3.1). Thus, we have

$$\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}} = \frac{1}{4}\|\nabla u\|^2_{L^2_{\mathrm{sc}}} + \frac{1}{4}\mathsf{F}^2\|u\|^2_{L^2_{\mathrm{sc}}} + \frac{3}{4}\|\delta^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}} + \langle Au, u\rangle + \langle \tilde{R}u, u\rangle.$$

This gives us

$$\|\nabla u\|^2_{L^2_{\mathrm{sc}}} + \mathsf{F}^2\|u\|^2_{L^2_{\mathrm{sc}}} \leqslant C\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}} + C\|x^{1/2}u\|^2_{L^2_{\mathrm{sc}}} + C\|u\|_{\dot{H}^{1,0}_{\mathrm{sc}}}\|u\|_{L^2_{\mathrm{sc}}} + C\mathsf{F}\|u\|^2_{L^2_{\mathrm{sc}}}. \tag{3.5}$$

Then for sufficiently large F,

$$\|\nabla u\|^2_{L^2_{\mathrm{sc}}} + \mathsf{F}^2\|u\|^2_{L^2_{\mathrm{sc}}} \leqslant C\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}} + C\|x^{1/2}u\|^2_{L^2_{\mathrm{sc}}}, \tag{3.6}$$

where C is a constant depending on F, and thus

$$\|\nabla u\|^2_{L^2_{\mathrm{sc}}} + \langle (1 - Cx)u, u\rangle \leqslant C\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}}.$$

Now suppose that Ω_j is contained in $\{x \leqslant x_0\}$. If x_0 is sufficiently small, this gives

$$\|\nabla u\|_{L^2_{\mathrm{sc}}} + \|u\|_{L^2_{\mathrm{sc}}} \leqslant C\|\mathrm{d}^s_{\mathsf{F}} u\|_{L^2_{\mathrm{sc}}}. \tag{3.7}$$

If x_0 is larger, we can still have

$$\|\nabla u\|_{L^2_{\mathrm{sc}}} + \|u\|_{L^2_{\mathrm{sc}}} \leqslant C\|\mathrm{d}^s_{\mathsf{F}} u\|_{L^2_{\mathrm{sc}}} + C\|u\|_{L^2_{\mathrm{sc}}(\{x_1 \leqslant x \leqslant x_0\})},$$

with x_1 small, and thus have (3.7) by the standard Poincaré inequality (see [14, equation (28)] for one forms). Then, with (3.3), and choosing $\epsilon > 0$ small, we find that

$$\|u\|_{\dot{H}^{1,0}_{\mathrm{sc}}} \leqslant C\|\Delta_{\mathsf{F},s}u\|_{\bar{H}^{-1,0}_{\mathrm{sc}}}.$$

Therefore, we have proved the invertibility of $\Delta_{\mathsf{F},s}$. □

Using lemma 4.4 in [16], in parallel with the above lemmas, we obtain:

Lemma 3.3. *There exists* $\mathsf{F}_0 > 0$ *such that for* $\mathsf{F} > \mathsf{F}_0$, *the operator* $\Delta_{\mathsf{F},s} = \delta^s_{\mathsf{F}} d^s_{\mathsf{F}}$ *on symmetric three-tensors is invertible as a map* $\dot{H}^{1,r}_{\mathrm{sc}} \to \bar{H}^{-1,r}_{\mathrm{sc}}$ *for all* $r \in \mathbb{R}$.

Lemma 3.4. *Let* Ω_j *be a domain contained in X as above. For* $\mathsf{F} > 0$ *and* $r \in \mathbb{R}$,

$$\|u\|_{\bar{H}^{1,r}_{\mathrm{sc}}(\Omega_j)} \leqslant C(\|x^{-r} d^s_{\mathsf{F}} u\|_{L^2_{\mathrm{sc}}(\Omega_j)} + \|u\|_{x^{-r}L^2_{\mathrm{sc}}(\Omega_j)}),$$

for symmetric three-tensors $u \in \bar{H}^{1,r}_{\mathrm{sc}}(\Omega_j)$.

Proof. By the proof of lemma 4.5 in [16], we only need to consider the case $r = 0$. Let $\tilde{\Omega}_j$ be a domain in X with C^∞ boundary, transversal to ∂X, containing $\overline{\Omega_j}$. We show that there exists a continuous extension map $E : \bar{H}^{1,2}_{\mathrm{sc}}(\Omega_j) \to \dot{H}^{1,2}_{\mathrm{sc}}(\tilde{\Omega}_j)$ such that

$$\|d^s_{\mathsf{F}} Eu\|_{L^2_{\mathrm{sc}}(\tilde{\Omega}_j)} + \|Eu\|_{L^2_{\mathrm{sc}}(\tilde{\Omega}_j)} \leqslant C(\|d^s_{\mathsf{F}} u\|_{L^2_{\mathrm{sc}}(\Omega_j)} + \|u\|_{L^2_{\mathrm{sc}}(\Omega_j)}),\ u \in \bar{H}^{1,0}_{\mathrm{sc}}(\Omega_j). \tag{3.8}$$

Once (3.8) is proved, by lemma 3.1, with $v = Eu$, we have

$$\begin{aligned}\|\nabla v\|^2_{L^2_{\mathrm{sc}}(\tilde{\Omega}_j)} + \|v\|^2_{L^2_{\mathrm{sc}}(\tilde{\Omega}_j)} &\leqslant C(\|d^s_{\mathsf{F}} v\|^2_{L^2_{\mathrm{sc}}(\tilde{\Omega}_j)} + \|v\|^2_{L^2_{\mathrm{sc}}(\tilde{\Omega}_j)}) \\ &\leqslant C(\|d^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}(\Omega_j)} + \|v\|^2_{L^2_{\mathrm{sc}}(\Omega_j)}).\end{aligned}$$

This finally gives

$$\|u\|_{\bar{H}^{1,0}_{\mathrm{sc}}(\Omega_j)} \leqslant C(\|d^s_{\mathsf{F}} u\|^2_{L^2_{\mathrm{sc}}(\Omega_j)} + \|v\|^2_{L^2_{\mathrm{sc}}(\Omega_j)}).$$

It only remains to construct E. By a partition of unity, this can be reduced to the situation where locally $X = \overline{\mathbb{R}^n}$, $\overline{\Omega_j} = \overline{\mathbb{R}^n_+}$; see the proof of lemma 4.5 in [16]. We only need to analyze the extension of a symmetric 3-tensor on $\overline{\mathbb{R}^n_+}$ to $\overline{\mathbb{R}^n}$.

We let $\Phi_q(x', x'_n) = (x', -qx_n)$ for $x_n < 0$ be a diffeomorphism from $\{x_n < 0\}$ to $\{x_n > 0\}$. For $f_{ijk} dx^i \otimes dx^j \otimes dx^k$ on $\{x_0 \geqslant 0\}$, we define E_1 to be the extension to R^n,

$$E_1(f_{ijk} dx^i \otimes dx^j \otimes dx^k)(x', x_n) = \sum_{q=1}^{5} C_q \Phi_q^*(f_{ijk} dx^i \otimes dx^j \otimes dx^k),\ x_n < 0$$

and

$$E_1(f_{ijk} dx^i \otimes dx^j \otimes dx^k)(x', x_n) = f_{ijk} dx^i \otimes dx^j \otimes dx^k,\ x_n \geqslant 0,$$

with C_q chosen so that $E_1 : C^1(\overline{\mathbb{R}^n_+}) \to C^1(\overline{\mathbb{R}^n})$. By calculation

$$
\begin{aligned}
&\Phi_q^* f_{ijk} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = f_{ijk}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \; i,j,k \neq n, \\
&\Phi_q^* f_{ijn} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n = -q f_{ijn}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n, \; i,j \neq n, \\
&\Phi_q^* f_{inn} \mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = q^2 f_{inn}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \; i \neq n, \\
&\Phi_q^* f_{nnn} \mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = -q^3 f_{nnn}(x', -qx_n) \mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \\
&\partial_l \Phi_q^* f_{ijk} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = \partial_l f_{ijk}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \; i,j,k,l \neq n, \\
&\partial_l \Phi_q^* f_{ijn} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n = -q \partial_l f_{ijn}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n, \; i,j,l \neq n, \\
&\partial_l \Phi_q^* f_{inn} \mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = q^2 \partial_l f_{inn}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \; i,l \neq n, \\
&\partial_l \Phi_q^* f_{nnn} \mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = -q^3 \partial_l f_{nnn}(x', -qx_n) \mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \; l \neq 0 \\
&\partial_n \Phi_q^* f_{ijk} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = -q \partial_n f_{ijk}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \; i,j,k \neq n, \\
&\partial_n \Phi_q^* f_{ijn} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n = q^2 \partial_n f_{ijn}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n, \; i,j \neq n, \\
&\partial_n \Phi_q^* f_{inn} \mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = -q^3 \partial_n f_{inn}(x', -qx_n) \mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \; i \neq n, \\
&\partial_n \Phi_q^* f_{nnn} \mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = q^4 \partial_n f_{nnn}(x', -qx_n) \mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n.
\end{aligned}
$$

The matching of the derivatives at $x_n = 0$, which gives the C^1 property, yields

$$
\begin{aligned}
&C_1 + C_2 + C_3 + C_4 + C_5 = 1, \\
&C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 = -1, \\
&C_1 + 4C_2 + 9C_3 + 16C_4 + 25C_5 = 1, \\
&C_1 + 8C_2 + 27C_3 + 64C_4 + 125C_5 = -1, \\
&C_1 + 16C_2 + 81C_3 + 256C_4 + 625C_5 = 1.
\end{aligned}
$$

The linear system, with a Vandermonde matrix, is solvable. With the $C_q, q = 1, 2, \cdots, 5$, satisfying the linear system above, we obtain the property $E_1 : C^1_c(\overline{\mathbb{R}^n_+}) \to C^1_c(\overline{\mathbb{R}^n})$ and

$$\|E_1 u\|_{H^1(\mathbb{R}^n)} \leqslant C \|u\|_{H^1(\mathbb{R}^n_+)}.$$

With Φ_q^* acting on four-tensors as usual, we have

$$\mathrm{d}^s \Phi_q^* = \Phi_q^* \mathrm{d}^s,$$

and thus

$$\|\mathrm{d}^s \Phi_q^* u\|_{L^2(\mathbb{R}^n_-)} \leqslant C \|\mathrm{d}^s u\|_{L^2(\mathbb{R}^n_+)}.$$

Then

$$\|\mathrm{d}^s E_1 u\|_{L^2(\mathbb{R}^n)} \leqslant C \|\mathrm{d}^s u\|_{L^2(\mathbb{R}^n_+)}$$

which completes the proof. □

We define

$$
\begin{aligned}
&\mathcal{S}_{\mathsf{F},\Omega_j} = \mathrm{Id} - \mathrm{d}^s_{\mathsf{F}} \Delta^{-1}_{\mathsf{F},s} \delta^s_{\mathsf{F}}, \\
&\mathcal{P}_{\mathsf{F},\Omega_j} = \mathrm{d}^s_{\mathsf{F}} \mathcal{Q}_{\mathsf{F},\Omega_j}, \; \mathcal{Q}_{\mathsf{F},\Omega_j} = \Delta^{-1}_{\mathsf{F},s} \delta^s_{\mathsf{F}}.
\end{aligned}
$$

We have the following corollaries for the properties of Dirichlet Laplacian $\Delta_{\mathsf{F},s}$ in parallel with corollaries 4.6–4.8 in [16]:

Corollary 3.5. *Let ϕ on $C_c^\infty(\overline{\Omega_j}\setminus\partial_{\mathrm{int}}\Omega_j)$. Then on symmetric three-tensors, there exists $\mathsf{F}_0>0$ such that for any $\mathsf{F}\geqslant\mathsf{F}_0$, $\phi\Delta_{\mathsf{F},s}^{-1}\phi:\bar{H}_{\mathrm{sc}}^{-1,k}\to\dot{H}_{\mathrm{sc}}^{1,k}$ is in $\Psi_{\mathrm{sc}}^{-2,0}(X)$.*

Corollary 3.6. *Let $\phi\in C_c^\infty(\overline{\Omega_j}\setminus\partial_{\mathrm{int}}\Omega_j)$, $\chi\in C^\infty(\overline{\Omega_j})$ with disjoint support and with χ constant near $\partial_{\mathrm{int}}\Omega_j$. Let F,F_0 as in corollary . Then the operator $\chi\Delta_{\mathsf{F},s}^{-1}\phi:\bar{H}_{\mathrm{sc}}^{-1,k}(\Omega_j)\to\dot{H}_{\mathrm{sc}}^{1,k}(\Omega_j)$ in fact maps $H_{\mathrm{sc}}^{s,r}(X)\to\dot{H}_{\mathrm{sc}}^{1,k}(\Omega_j)$ for all s,r,k.*

Similarly, $\phi\Delta_{\mathsf{F},s}^{-1}\chi:\bar{H}_{\mathrm{sc}}^{-1,k}(\Omega_j)\to\dot{H}_{\mathrm{sc}}^{1,k}(\Omega_j)$ in fact maps $\bar{H}_{\mathrm{sc}}^{-1,k}(\Omega_j)\to H_{\mathrm{sc}}^{s,r}(X)$ for all s,r,k.

Corollary 3.7. *Let $\phi\in C_c^\infty(\overline{\Omega_j}\setminus\partial_{\mathrm{int}}\Omega_j)$, $\chi\in C^\infty(\overline{\Omega_j})$ with disjoint support and with χ constant near $\partial_{\mathrm{int}}\Omega_j$. Let F,F_0 as in corollary .*

Then $\phi\mathcal{S}_{\mathsf{F},\Omega_j}\phi\in\Psi_{\mathrm{sc}}^{0,0}(X)$, while $\chi\mathcal{S}_{\mathsf{F},\Omega_j}\phi:H_{\mathrm{sc}}^{s,r}(X)\to x^kL_{\mathrm{sc}}^2(\Omega_j)$ and $\phi\mathcal{S}_{\mathsf{F},\Omega_j}\chi:x^kL_{\mathrm{sc}}^2(\Omega_j)\to H_{\mathrm{sc}}^{s,r}(X)$ for all s,r,k.

Then we can proceed as follows. Let Ω_2 be a larger neighborhood of Ω. Let G be a parametrix of A_{F} in Ω_2, and it is a scattering pseudodifferential operator. Then $GA_{\mathsf{F}}=I+E$, where $\mathrm{WF}'_{\mathrm{sc}}(E)$ is disjoint from a neighborhood $\Omega_1\subset\subset\Omega_2$ of Ω, and $E=-\mathrm{Id}$ near $\partial_{\mathrm{int}}\Omega_2$. Now we have

$$G(N_{\mathsf{F}}+\mathrm{d}_{\mathsf{F}}^sM\delta_{\mathsf{F}}^s)=I+E.$$

This yields

$$\mathcal{S}_{\mathsf{F},\Omega_2}G(N_{\mathsf{F}}+\mathrm{d}_{\mathsf{F}}^sM\delta_{\mathsf{F}}^s)\mathcal{S}_{\mathsf{F},\Omega_2}=\mathcal{S}_{\mathsf{F},\Omega_2}+\mathcal{S}_{\mathsf{F},\Omega_2}E\mathcal{S}_{\mathsf{F},\Omega_2}.$$

Notice that

$$N_{\mathsf{F}}\mathcal{S}_{\mathsf{F},\Omega_2}=N_{\mathsf{F}}$$

and

$$\delta_{\mathsf{F}}^sN_{\mathsf{F}}=0,$$

and we then obtain

$$\mathcal{S}_{\mathsf{F},\Omega_2}GN_{\mathsf{F}}=\mathcal{S}_{\mathsf{F},\Omega_2}+\mathcal{S}_{\mathsf{F},\Omega_2}E\mathcal{S}_{\mathsf{F},\Omega_2}.$$

In parallel with [16, lemma 4.10], we have the smallness of $K_1=\mathcal{S}_{\mathsf{F},\Omega_2}E\mathcal{S}_{\mathsf{F},\Omega_2}$. Denote r_{21} as the restriction map from Ω_2 to Ω_1 and e_{12} as the extension map from Ω_1 to Ω_2. Similar to [16, lemma 4.11], we can show the smallness of

$$\mathcal{S}_{\mathsf{F},\Omega_1}-r_{21}\mathcal{S}_{\mathsf{F},\Omega_2}e_{12}.$$

Then similar to [16, (4.12)], we have

$$\mathcal{S}_{\mathsf{F},\Omega_1}r_{21}\mathcal{S}_{\mathsf{F},\Omega_2}GN_{\mathsf{F}}=\mathcal{S}_{\mathsf{F},\Omega_1}e_{01}+K_2,$$

where K_2 is smoothing and small in $\Omega\subset\{x\leqslant\delta\}$ with δ sufficiently small. Here e_{01} is the extension map from Ω to Ω_1, and r_{10} is the restriction map from Ω_1 to Ω. Restricting to Ω from the left, we get

$$r_{10}\mathcal{S}_{\mathsf{F},\Omega_1}r_{21}\mathcal{S}_{\mathsf{F},\Omega_2}GN_{\mathsf{F}}=r_{10}\mathcal{S}_{\mathsf{F},\Omega_1}e_{01}+r_{10}K_2.$$

Proceeding as [16], we can analyze $r_{10}\mathcal{S}_{\mathsf{F},\Omega_1}e_{01}-\mathcal{S}_{\mathsf{F},\Omega}$, and arrive at the main, local, result.

Theorem 3.8. *For $\Omega = \Omega_c$, $c > 0$ small, there exists $\mathsf{F}_0 > 0$ large enough, such that for $\mathsf{F} > \mathsf{F}_0$, the geodesic ray transform on symmetric four-tensors $f \in \mathrm{e}^{\mathsf{F}/x} L^2_{\mathrm{sc}}(\Omega)$ satisfying $\delta^s(\mathrm{e}^{-2\mathsf{F}/x} f) = 0$ is injective with a stability estimate. Here the stability is in the sense that for $s \geqslant 0$, there exists R, R' such that for any sufficiently negative r, the $\mathrm{e}^{\mathsf{F}/x} H^{s-1,r}_{\mathrm{sc}}$ norm of f on Ω is controlled by the $\mathrm{e}^{\mathsf{F}/x} H^{s,r+R}_{\mathrm{sc}}$ norm of $I_4 f$, provided that f is a priori in $\mathrm{e}^{\mathsf{F}/x} H^{s,r+R'}_{\mathrm{sc}}$.*

A version of the above theorem using local Sobolev spaces is stated in theorem 1.1. The theorem also leads to the global result, by a layer stripping scheme, theorem 1.2 similar to [16, theorem 4.19].

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