LAGRANGE MULTIPLIERS AND CONSTRAINED MINIMIZING PROGRAM

The material of these notes is from

Numerical Optimization, Springer (2006),
by **Jorge Nocedal** and **Stephen J. Wright**.

Through out this section we assume that f is a smooth real valued function defined in $\mathbf{R}^n$. Recall that the unconstrained minimization program is

$$\min_{x\in\mathbf{R}^n} f(x). \tag{1}$$

We have learned that the following conditions hold:

(N) If $x^* \in \mathbf{R}^n$ is a local solution of (1) then

$$\nabla f(x^*) = 0, \quad \text{and} \quad \nabla^2 f(x^*) \text{ is positive semidefinite.}$$

(S) If there exists $x^* \in \mathbf{R}^n$ such that

$$\nabla f(x^*) = 0, \quad \text{and} \quad \nabla^2 f(x^*) \text{ is positive definite,}$$

then x^* is a local solution of (1).

These are called *Necessary* and *Sufficient* optimality conditions. For instance Newtons method is an iterative way of solving this problem.

Let $\mathcal{I} \cup \mathcal{E}$ be some distinct index sets. We call functions smooth functions c_i, $i \in \mathcal{I} \cup \mathcal{E}$ the constrains. The corresponding minimization program

$$\begin{cases} \min\limits_{x\in\mathbf{R}^n} f(x) \\ c_i(x) = 0, \quad i \in \mathcal{I} \\ c_j(x) \geq 0, \quad j \in \mathcal{E}. \end{cases} \tag{2}$$

is called the *Constrained minimization program*. The set

$$\Omega = \{x \in \mathbf{R}^n : \ c_i(x) = 0, \ i \in \mathcal{I}, \ c_j(x) \geq 0, \ j \in \mathcal{E}\}$$

is called the *feasible set* . We note that (2) is equivalent to

$$\min_{x\in\Omega} f(x).$$

Observe that Ω is typically a domain in some lower dimensional space $\mathbf{R}^k$, $k \leq n$. Therefore to solve (2) we cannot just imply the previous optimality conditions. However we do the following observation. Suppose that $x \in \Omega$ and $s \in \mathbf{R}^n$ are such that $s + x \in \Omega$. If

$$\nabla f(x) \cdot s < 0, \tag{3}$$

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then the Taylor expansion

$$f(x+s) = f(x) + \nabla f(x) \cdot s + \mathcal{O}(\|s\|)$$

imply that x is not a local solution of (2). On the other hand for the constrained c_i, $i \in \mathcal{I}$ the Taylor expansion

$$c_i(x+s) = c_i(x) + \nabla c_i(x) \cdot s + \mathcal{O}(\|s\|) \quad \Leftrightarrow \quad 0 = \nabla c_i(x) \cdot s + \mathcal{O}(\|s\|),$$

suggests

$$\nabla c_i(x) \cdot s = 0. \tag{4}$$

Therefore if there is no $s \in \mathbf{R}^n$ for which (4) holds but for which (3) does not hold then x could be a local solution of (2).

We note that if

$$\nabla c_i(x) = \lambda \nabla f(x), \qquad \text{for some } \lambda \in \mathbf{R},$$

Then (3) and (4) cannot hold simultaneously for any $s \in \mathbf{R}^n$. Conversely if $\nabla c_i(x)$ and $\nabla f(x)$ are not parallel then the choice

$$s = \lambda \left(\frac{\nabla c_i(x) \nabla c_i(x)^T}{\|\nabla c_i(x)\|_2^2} - I \right) \nabla f(x), \quad \lambda > 0$$

will satisfy both (3) and (4).

For a given $i \in \mathcal{I}$ we call the function

$$\mathcal{L}(x, \lambda) := f(x) - \lambda c_i(x), \quad (x, \lambda) \in \mathbf{R}^{n+1}$$

as the *Langrangian function.* Thus in order to solve (2) it makes sense to study the stationary points of $\mathcal{L}$ i.e. those $(x^*, \lambda^*) \in \mathbf{R}^{n+1}$ for which

$$\nabla_{x,\lambda} \mathcal{L}(x^*, \lambda^*) = 0 \quad \Leftrightarrow \quad \begin{cases} \nabla f(x^*) = \lambda^* \nabla c_i(x^*) \\ c_i(x^*) = 0 \end{cases} \tag{5}$$

If $(x^*, \lambda^*) \in \mathbf{R}^{n+1}$ solves (5) then λ^* is called the *Lagrangian multiplier at* x^*.

The condition may be (5) might be necessary but it is not sufficient!

Example 1. Let $f(x) = x_1 + y_2$ and $c(x) = \|x\|_2^2 - 1$. Then

$$\min_{x \in c^{-1}\{0\}} f(x)$$

has a solution $(-1, -1)$ and

$$\nabla f(-1,-1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \nabla c(-1,-1) = -2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

on the other hand

$$\nabla f(1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \nabla c(1,1) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

but $(1, 1)$ is max

To solve the minimization program we define for every $x \in \Omega$ an active set of constrains

$$A(x) = \mathcal{I} \cup \{ j \in \mathcal{E} : c_j(x) = 0 \}.$$

Definition 1. *We say that at a point $x \in \Omega$ the Linear Independence Constraint Qualification (LICQ) holds if*

$$\{\nabla c_i(x) : i \in A(x)\} \text{ is a linearly independent set.}$$

The main theorem we aim the prove is the following:

Theorem 2. *We define a Lagrangian function of minimization program* (2) *as*

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{I} \cup \mathcal{E}} \lambda_i c_i(x).$$

If $x^* \in \Omega$ solves (2) *and (LICQ) holds at x^* then the following holds:*

- *There exists a Lagrangian vector λ^* such that*

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \quad \text{for all } i \in \mathcal{I} \cup \mathcal{E}$$

- $\lambda_i \geq 0$ *for all* $i \in \mathcal{E}$.

- $\lambda_i^* c_i(x^*) = 0, \quad$ *for all* $i \in \mathcal{I} \cup \mathcal{E}$.

Remark 1. If there is only one constraint c and $\Omega = c^{-1}\{0\}$, then gradients of f and c are parallel at the minimizer.

To prove this theorem we give two definitions.

Definition 3 (Tangent cone)**.** *The vector d is said to be a tangent (or tangent vector) to Ω at a point $x \in \Omega$ if there are a feasible sequence $(z_k)_{k=1}^{\infty} \in \Omega$ approaching x and a sequence of positive scalars $(t_k)_{k=1}^{\infty}$ with $t_k \to 0$ such that*

$$d = \lim_{k \to \infty} \frac{z_k - x}{t_k}.$$

The set of all tangents to Ω at x is called the tangent cone and is denoted by $T_x\Omega$.

Almost immediate consequence of the previous definition is the following necessity theorem.

Theorem 4 (Necessary conditions)**.** *If $x^* \in \Omega$ is a solution of* (2)*, then*

$$\nabla f(x^*) d \geq 0, \quad \text{for all} d \in T_X \omega.$$

Proof. Assume that the claim does not hold for some $d \in T_x\Omega$. Then prove that x^* cannot be a local minimina by using the definition of $T_x\Omega$ and the Taylor expansion of f at x. ☐

Definition 5 (LFD)**.** *Given a feasible point $x \in \Omega$ and the active constraint set $A(x)$ the set of Linearized Feasible Directions (LFD)*

$$\mathcal{F}(x) := \{d \in \mathbf{R}^n : \nabla c_i(x) \cdot d = 0,\ i \in \mathcal{I},$$
$$\nabla c_j(x) \cdot d \geq 0,\ j \in \mathcal{E} \cap A(x)\}.$$

The first steps to prove Theorem 2 is the following lemma.

Lemma 1. *If (LICG) holds at $x \in \Omega$, then*

$$T_x\Omega = \mathcal{F}(x).$$

Proof. Let $d \in T_x\Omega$. Choose a sequence $(z_k)_{k=1}^{\infty} \in \Omega$ approaching x and a sequence of positive scalars $(t_k)_{k=1}^{\infty}$ with $t_k \to 0$ that satisfy

$$d = \lim_{k\to\infty} \frac{z_k - x}{t_k} \quad \Leftrightarrow \quad z_k = t_k d + x + h(t_k)t_k, \text{ for some function } h(t) \to 0,\ t \to 0.$$

Therefore the Taylor expansion of c_i, $i \in A(x)$ at x implies

$$0 \leq \frac{1}{t_k} c_i(z_k) = \frac{1}{t_k}(t_k \nabla c_i \cdot d + \mathcal{O}(t_k)) = \nabla c_i \cdot d + h(t_k) \quad \overset{k\to\infty}{\Rightarrow} \quad \nabla c_i \cdot d \geq 0.$$

And the $\geq$ can be replaced with $=$ if $i \in \mathcal{I}$. This implies

$$d \in \mathcal{F}(x).$$

Let $d \in \mathcal{F}(x)$. We use the notation

$$M(x) = \text{ for the } m \times n \text{ matrix whose rows are } \nabla c_i(x)^T,\ i \in A(x).$$

Due to $(LICG)$ the rank of $M(x) = m \leq n$. Thus $M(x)$ has a kernel of dimension $n - m$. Let Z be matrix whose columns form a basis of $\ker A(x)$, that is

$$Z \in \mathbf{R}^{n\times(n-m)}, \quad Z \text{ has a full rank}, \quad M(x)Z = 0.$$

We will denote

$$c(z) := \begin{pmatrix} c_1(z), \\ \vdots \\ c_m(z), \end{pmatrix}$$

and define a vector valued map

$$\mathbf{R}^{n+1} \ni (z,t) \mapsto R(z,t) := \begin{pmatrix} c(z) - tM(x)d \\ Z^T(z - x - td) \end{pmatrix} \in \mathbf{R}^m \times \mathbf{R}^{n-m} = \mathbf{R}^n.$$

Then

$$R(x,0) = 0, \quad \nabla_z R(z,0)\Big|_{z=x} = \begin{pmatrix} M(x) \\ Z^T \end{pmatrix} \in \mathbf{R}^{n\times n}.$$

Moreover the block matrix $\begin{pmatrix} M(x) \\ Z^T \end{pmatrix}$ is non singular since

$$\begin{pmatrix} M(x) \\ Z^T \end{pmatrix} v = 0 \ \Rightarrow\ v \in \ker A \text{ and } v \cdot Z_i = 0 \text{ for all basis elements } Z_i \text{ of } \ker A \ \Rightarrow\ v = 0.$$

Therefore the implicit function theorem implies that there is a smooth function g defined from a neighborhood $U \subset \mathbf{R}$ of 0 onto neighborhood $V \subset \mathbf{R}^N$ of x such that

$$g(0) = x$$

and

$$R(z,t) = 0 \quad \Leftrightarrow \quad z = g(t), \quad z \in V, \quad t \in V.$$

That is

$$R(f(t),t) = 0, \quad \text{for all } t \in V.$$

Choose a positive sequence $(t_k)_{k=1}^{\infty} \subset U$ that converges to 0. Denote $z_k = f(t_k)$. Then

$$R(z_k, t_k) = 0 \text{ and } \quad d \in \mathcal{F}(x) \quad \Rightarrow \quad z_k \in \Omega.$$

Then we deduce that

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = \dot{f}(0)$$

and due to the Taylor expansion of c we obtain

$$0 = R(z_k, t_k) = \begin{pmatrix} c(z_k) - t_k M(x) d \\ Z^T(z_k - x - t_k d) \end{pmatrix} = \begin{pmatrix} M(x) \\ Z^T \end{pmatrix} (z_k - x - t_k d) + H(t_k)\|z_k - x\|,$$

for some vector valued function H that satisfies

$$H(t) \to 0, \quad t \to 0.$$

This imply

$$d = \frac{z_k - x}{t_k} + \begin{pmatrix} M(x) \\ Z^T \end{pmatrix}^{-1} H(t_k) \left\| \frac{z_k - x}{t_k} \right\|.$$

Therefore we conclude

$$d = \lim_{t\to\infty} \frac{z_k - x}{t_k} \in T_x\Omega.$$

This ends the proof. □

We still need one more technical lemma to prove Theorem 2. If $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{n\times h}$ then a set

$$K := \{By + Cw \in \mathbf{R}^n : y \in \mathbf{R}^m,\ y_i \geq 0,\ w \in \mathbf{R}^h\}$$

is a cone, i.e. it satisfies

$$v \in K \quad \Rightarrow \quad tv \in K,\ t \geq 0.$$

The lemma we need is the Fargas lemma:

Lemma 2 (Fargas)**.** *Let $g \in \mathbf{R}^n$, then exactly one of the following holds.*

(i) $g \in K$

(ii) *there exists a vector in $d \in \mathbf{R}$*

$$g \cdot d < 0, \quad \textit{every component of } dB \textit{ is non negative}, \quad dC = 0.$$

Proof. We show first that the both conditions cannot hold simultaneously. If so then there exist vectors $y \in \mathbf{R}^m$, $y_i \geq 0$ and $w \in \mathbf{R}^h$ such that

$$g = By + Cw.$$

Then the second condition implies the existence of $d \in \mathbf{R}^n$ so that

$$0 > g \cdot d = d \cdot (By + Cw) = (dB)y + (dC)w = (dB)y = \sum_{i=1}^{m} (dC)_i y_i \geq 0.$$

And we arrive to a contradiction.

To prove that exactly one of the previous conditions holds we assume that $g \neq K$. Then we note that K is a closed set, i.e. it contains all of its accumulation points:

If x is an accumulation point of K then there exists a sequence $x_k \in K$ that converges to x. For every $k \in \mathbf{N}$ we choose y_k, w_k such that

$$z_k = By_k + Cw_k. \tag{6}$$

Let $\overline{(y_k, z_k)}$ be a least square solution of (6). Then sequence $\overline{(y_k, z_k)}$ is bounded in $\mathbf{R}^{n+h}$. Therefore it has a convergent sub sequence. If (y, z) is this limit then the continuity of linear map B, C implies that

$$x = By + Cw \in K.$$

Since K is closed there exists a closest point $x \in K$ to g and moreover

$$\|g - x\|_2 > 0.$$

Since K is conic we have for any $t \geq 0$ that

$$tx \in K, \quad \phi(t) := \|g - tx\|_2^2 \geq \|g - x\|_2^2 \quad \Rightarrow \quad 0 = \frac{d}{dt}\phi(t)_{t=1} = 2x \cdot (x - g).$$

Since K it convex it holds for any $z \in K$ and $t \in [0, 1]$ that

$$\|x + t(z - x) - g\|_2^2 \geq \|x - g\|_2^2 \quad \Rightarrow \quad 2t(z - x) \cdot (x - g) + t^2\|z - x\|_2^2 \geq 0.$$

This implies

$$(z - x) \cdot (x - g) = z \cdot (x - g) \geq 0, \quad \text{for all } z \in K.$$

Finally we define

$$d := x - g \neq 0.$$

Then we have

$$d \cdot g = d \cdot (x - d) = -\|d\|_2^2 < 0.$$

Also

$$d \cdot (By + Cw) \geq 0, \quad \text{forall } y,\ y_i \geq 0, \text{ and } w.$$

This implies

$$(dB) \cdot y \geq 0, \text{ as } y_i \geq 0 \quad \Rightarrow \quad (dB)_i \geq 0,$$

and

$$(dC) \cdot w \geq 0 \text{ for all } w \in \mathbf{R}^h \quad \Rightarrow \quad dC = 0.$$

Therefore the claim is proven. □

Finally we prove Theorem 2.

Proof of Theorem 2. We define a cone

$$K = \left\{ \sum_{i \in A(x)} t_i \nabla c_i(x^*) \in \mathbf{R}^n : t_i \in \mathbf{R},\ t_i \geq 0 \text{ if } i \in A(x) \cap \mathcal{E}. \right\}$$

Then

$$K = \{By + Cw \in \mathbf{R}^n : y \in \mathbf{R}^m,\ y_i \geq 0,\ w \in \mathbf{R}^h\},$$

where $B \in \mathbf{R}^{n\times m}$ is the matrix with columns $\nabla c_i(x^*)$, $i \in A(x) \cap \mathcal{E}$ and $C \in \mathbf{R}^{n\times h}$ is the matrix with columns $\nabla c_i(x^*)$, $i \in \mathcal{I}$. Now it either holds that

$$\nabla f(x^*) \in K \quad \Rightarrow \quad \nabla f(x^*) = \sum_{i\in A(x)} \lambda_i \nabla c_i(x^*), \ \text{ for some } \lambda_i \in \mathbf{R}.$$

or

$$\text{there exists a vector } d \in \mathcal{F}(x^*) \quad \text{such that } d \cdot \nabla f(x^*) < 0.$$

Since we have proved that

$$\mathcal{F}(x^*) = T_{x^*}\Omega,$$

The Theorem 4 of Necessary optimal conditions exclude the latter option.

Therefore

$$\nabla f(x^*) = \sum_{i\in A(x)} \lambda_i \nabla c_i(x^*) \text{ for some } \lambda_i \in \mathbf{R}, \quad \lambda_i \geq 0,\ i \in A(x) \cap \mathcal{E}. \tag{7}$$

If we define a vector

$$\lambda_i^* = \lambda_i, \text{ if } i \in A(x) \text{ and othervice } (\lambda^*)_i = 0.$$

Then (7) implies the claim

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0.$$

On the other hand (7) implies

$$\lambda_i^* \geq 0 \quad \text{for all } i \in \mathcal{E}.$$

The final claim

$$\lambda_i c_i(x^*) = 0$$

follows from the definition of the active set $A(x)$ and λ^*. □

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