

CAAM 423/523, MATH 423/513 PROBLEM SET 5

The solutions to the following exercises should be submitted to Xiao Liu's office at 2033 Duncan Hall by **4 pm Wednesday, 31 October 2018**. If he is not in, slip your work under the door. Note: This instruction may change in future problem sets depending on whether Prof. de Hoop or Dr. Wong is currently in town.

You may collaborate and discuss with other Rice students about non-pledged problems, but you must write your own solutions. Solutions should either be **typed** (in Latex) or **written very neatly**.

You **may not** consult any solution manuals, online forums, course materials from other courses, including courses at either Rice or other schools, or any other sources that may be viewed as outside assistance.

Where applicable, the following exercises use the notation from Evans' textbook.

PROBLEM 1

Let U be bounded with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$, $1 \leq p < \infty$, does not have a trace on ∂U . More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \rightarrow L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

PROBLEM 2

Integrate by parts to prove the interpolation inequality

$$\|Du\|_{L^2} \leq C \|u\|_{L^2}^{1/2} \|D^2u\|_{L^2}^{1/2},$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.

Hint: Take sequences $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(\bar{U})$ converging to u in $H^2(U)$.

PROBLEM 3

Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$ such that u is *not* Lipschitz continuous on U .

Hint: Take U to be the open unit disk in \mathbb{R}^2 , with a slit removed.

PROBLEM 4

This problem is pledged. You cannot discuss this problem with anyone other than the instructor, and you should not consult with any sources other than the textbook and your own notes.

Provide a counterexample showing the smooth approximation theorem fails for $p = \infty$. In particular, for some open set U , show that there exists $u \in W^{1,\infty}(U)$ such that there is no sequence $\{u_m\} \subset C^\infty(U) \cap W^{1,\infty}(U)$ satisfying $\|u - u_m\|_{W^{1,p}} \rightarrow 0$ as $m \rightarrow \infty$.