# CAAM 423/523, MATH 423/513 PROBLEM SET 3

The solutions to the following exercises should be submitted to Dr. Wong's office at 2037 Duncan Hall by **4 pm Friday, 5 October 2018**. If he is not in, either slip your work under the door or place it on the table (if door is open). Note: This instruction may change in future problem sets depending on whether Prof. de Hoop or Dr. Wong is currently in town.

You may collaborate and discuss with other Rice students about non-pledged problems, but you must write your own solutions. Solutions should either be **typed** (in Latex) or **written very neatly**.

You **may not** consult any solution manuals, online forums, course materials from other courses, including courses at either Rice or other schools, or any other sources that may be viewed as outside assistance.

Where applicable, the following exercises use the notation from Evans' textbook.

## Problem 1

(1) Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{cases} \mathbf{E}_t = \operatorname{curl} \mathbf{B}, \quad \mathbf{B}_t = -\operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{B} = \operatorname{div} \mathbf{E} = 0. \end{cases}$$

Show

$$\mathbf{E}_{tt} - \Delta \mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta \mathbf{B} = 0.$$

(2) Assume that $\mathbf{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\mathbf{u}_{tt} - \mu \Delta \mathbf{u} - (\lambda + \mu) D(\operatorname{div} \mathbf{u}) = \mathbf{0} \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show $w := \operatorname{div} \mathbf{u}$ and $\mathbf{w} := \operatorname{curl} \mathbf{u}$ each solve wave equations, but with differing speeds of propagation.

## Problem 2

Let $u$ solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose $g, h$ have compact support. The *kinetic energy* is

$$k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t)\, dx$$

and the *potential energy* is

$$p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t)\, dx.$$

Prove

(1) $k(t) + p(t)$ is constant in $t$,
(2) $k(t) = p(t)$ for all large enough times $t$.

## Problem 3

Use the method of characteristics to solve:

(1) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2)$.
(2) $u u_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2} x_1$.

## Problem 4

**This problem is pledged**. You cannot discuss this problem with anyone other than the instructor, and you should not consult with any sources other than the textbook and your own notes.

(1) Using the method of characteristics, confirm that the formula

$$u = g(x - tf'(u))$$

provides an implicit solution for the conservation law

$$\begin{cases} u_t + (f(u))_x = 0 & \mathbb{R} \times (0, \infty) \\ u = g & \mathbb{R} \times \{t = 0\}. \end{cases}$$

(2) When $f(u) = u^2/2$, the conservation law is called *Burger's equation.* Consider Burger's equation with initial condition

$$g(x) = \begin{cases} 0 & x < -1 \\ 1 + x & -1 \leq x < 0 \\ 1 - x & 0 \leq x < 1 \\ 0 & 1 \leq x. \end{cases}$$

Given this $f, g$, derive an explicit formula for $u(x,t)$ for $0 \leq t < 1$ using the implicit formula you derived above. Sketch the graph of the solution $u(x,t)$ at $t = 0, t = 1/2$, and $t$ approaching 1. Explain why the method of characteristics fails for constructing solutions for $t \geq 1$.

(Hint: $u(x,t)$ will be a continuous, piecewise-linear function in $x$ just like $g(x)$. Therefore it is in fact a *weak solution* that only solves Burger's equation outside a set of zero measure.)