CAAM 423/523, MATH 423/513 PROBLEM SET 2

The solutions to the following exercises should be submitted to Dr. Wong's office at 2037 Duncan Hall by **4 pm Friday, 28 September 2018**. If he is not in, either slip your work under the door or place it on the table (if door is open). Note: This instruction may change in future problem sets depending on whether Prof. de Hoop or Dr. Wong is currently in town.

You may collaborate and discuss with other Rice students about non-pledged problems, but you must write your own solutions. Solutions should either be **typed** (in Latex) or **written very neatly**.

You **may not** consult any solution manuals, online forums, course materials from other courses, including courses at either Rice or other schools, or any other sources that may be viewed as outside assistance.

Where applicable, the following exercises use the notation from Evans' textbook.

Problem 1

Give a direct proof (i.e. without using any mean value properties) that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

(Hint: Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\bar{U}_T$ at a point in U_T.)

Problem 2

We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T.$$

(1) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds$$

for all $E(x,t;r) \subset U_T$.

(2) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

(3) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove v is a subsolution.

(4) Prove $v := |Du|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.

Problem 3

(1) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G.

(2) Using the change of variables $\xi = x + t, \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(3) Use (1) and (2) to rederive d'Alembert's formula.

(4) Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?

Problem 4

This problem is pledged. You cannot discuss this problem with anyone other than the instructor, and you should not consult with any sources other than the textbook and your own notes.

Consider the inhomogeneous transport equation

$$\begin{cases} u_t + b \cdot Du = f & \text{on } \mathbb{R}^n \times [0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

where $b = b(x)$ is a function $\mathbb{R}^n \to \mathbb{R}^n$ such that there exists a constant $C > 0$ so that

$$|b| \le C, \quad |\operatorname{div}(b)| \le C$$

on $\mathbb{R}^n$. Furthermore, assume that $u(x,t)$ satisfies the decay condition that there exists $p > n/2$ such that for each $t \ge 0$, there exists $x_0 \in \mathbb{R}^n$ such that

$$|u(x,t)| \le C|x - x_0|^{-p}.$$

Using an energy method, prove that there exists at most one solution $u(\cdot, t) \in L^2(\mathbb{R}^n)$ to the above problem with the given decay condition.