CAAM 453: Numerical Analysis I

Problem Set 7: ODEs and LU
Due: Friday, December 1

Note: Turn in all MATLAB *code that you have written and turn in all output generated by your* MATLAB *functions/scripts.* MATLAB *functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.*

Problem 1: RK4 Stability (30 points)

In this problem, we will investigate the absolute stability region of RK4. Recall that RK4 is the following 4-stage method:

$$\begin{aligned}
k_1 &= f(x_k, t_k)\\
k_2 &= f(x_k + \tfrac{h}{2}k_1, t_k + \tfrac{h}{2})\\
k_3 &= f(x_k + \tfrac{h}{2}k_2, t_k + \tfrac{h}{2})\\
k_4 &= f(x_k + hk_3, t_k + h)\\
x_{k+1} &= x_k + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}$$

(a) When RK4 is applied to the model problem $x' = \lambda x$, $x(0) = x_0$, it gives the approximate solution $x_k = A(h\lambda)^k x_0$. Find the polynomial $A(h\lambda)$.

(b) Prove directly that the global error at any later time $t = c$ is $O(h^4)$. More precisely, show that if c/h is an integer, there exist constants M and h_0 such that

$$\left|x_{c/h} - x(c)\right| \le Mh^4 \qquad (*)$$

whenever $h \in (0, h_0]$, where $x(t)$ is the true solution of the initial value problem. You may use standard calculus theorems, but no theorems from this class.

Hint: Use your formula for $A(h\lambda)$ from part (a). One way of estimating the error $(*)$ is to use a Taylor series and the Mean Value Theorem.

(c) Using part (a) and Matlab, estimate the largest real number a such that $-a$ is in the stability region for RK4. Estimate the largest real number b such that ib is in the stability region.

Problem 2: Power Method for Non-Normal Matrices (30 points)

In this problem, we consider the behavior of the power method on non-diagonalizable matrices. If a square matrix A is not diagonalizable, it no longer has an associated orthonormal basis of eigenvectors. However, it does have an associated orthonormal basis of *generalized eigenvectors*. While an eigenvector is a non-zero vector in the nullspace of $A - \lambda I$ (where λ is the eigenvalue), a generalized eigenvector is a nonzero vector in the nullspace of $(A - \lambda I)^j$, for some integer $j > 0$.

We consider a simple example. Let A be an $n \times n$ real matrix having eigenvalues

$$\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > \cdots > \lambda_n > 0.$$

Assume we have an orthonormal set of vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ and a constant $\kappa \neq 0$ such that

$$\begin{aligned} Av_i &= \lambda_i v_i, && i \neq 2, \\ Av_2 &= \lambda_2 v_2 + \kappa v_1. \end{aligned}$$

That is, v_1 and $v_3, \ldots, v_n$ are eigenvectors of A, and v_2 is a generalized eigenvector.

Now, we try applying the power method to A. Let $w \in \mathbb{R}^n$ be a nonzero vector, and consider the sequence

$$w_k = \frac{A^k w}{\|A^k w\|}.$$

(a) Find $\lim_{k\to\infty} w_k$. Does the power method always converge to an eigenvector of A?

(b) Suppose that instead $\kappa = 0$, so that v_2 is in fact an eigenvector of A (and A is diagonalizable). Once again, find $\lim_{k\to\infty} w_k$. Does the power method always converge to an eigenvector of A?

Problem 3: Cholesky Factorization (40 points, pledged)

In class, we discussed the Cholesky factorization $M = LL^*$. Write MATLAB code to implement the Cholesky factorization for a Hermitian positive definite matrix $M \in \mathbb{C}^{n\times n}$. Your code should return an error if M is not Hermitian, or if some α is nonpositive, in which case M is not positive definite.

Test your code in the following way: Generate a number of random complex lower-triangular matrices L_0 with positive diagonal entries. For each, multiply $M = L_0 L_0^*$, and apply your Cholesky factorization algorithm to get an approximate factorization $M = LL^*$, then compute the error $\|L - L_0\|$. Report on the errors you find.