CAAM 453: Numerical Analysis I

Problem Set 7: ODEs and LU
Due: Friday, December 1

Note: Turn in all MATLAB code that you have written and turn in all output generated by your MATLAB functions/scripts. MATLAB functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.

Problem 1: RK4 Stability (30 points)

In this problem, we will investigate the absolute stability region of RK4. Recall that RK4 is the following 4-stage method:

\[
\begin{align*}
k_1 &= f(x_k, t_k) \\
k_2 &= f(x_k + \frac{h}{2}k_1, t_k + \frac{h}{2}) \\
k_3 &= f(x_k + \frac{h}{2}k_2, t_k + \frac{h}{2}) \\
k_4 &= f(x_k + hk_3, t_k + h) \\
x_{k+1} &= x_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{align*}
\]

(a) When RK4 is applied to the model problem \(x' = \lambda x, \ x(0) = x_0\), it gives the approximate solution \(x_k = A(h\lambda)^kx_0\). Find the polynomial \(A(h\lambda)\).

(b) Prove directly that the global error at any later time \(t = c\) is \(O(h^4)\). More precisely, show that if \(c/h\) is an integer, there exist constants \(M\) and \(h_0\) such that

\[|x_{c/h} - x(c)| \leq Mh^4 \quad (*)\]

whenever \(h \in (0, h_0]\), where \(x(t)\) is the true solution of the initial value problem. You may use standard calculus theorems, but no theorems from this class.

*Hint: Use your formula for \(A(h\lambda)\) from part (a). One way of estimating the error \((*)\) is to use a Taylor series and the Mean Value Theorem.*

(c) Using part (a) and Matlab, estimate the largest real number \(a\) such that \(-a\) is in the stability region for RK4. Estimate the largest real number \(b\) such that \(ib\) is in the stability region.

Problem 2: Power Method for Non-Normal Matrices (30 points)

In this problem, we consider the behavior of the power method on non-diagonalizable matrices. If a square matrix \(A\) is not diagonalizable, it no longer has an associated orthonormal basis of eigenvectors. However, it does have an associated orthonormal basis of generalized eigenvectors. While an eigenvector is a non-zero vector in the nullspace of \(A - \lambda I\) (where \(\lambda\) is the eigenvalue), a generalized eigenvector is a nonzero vector in the nullspace of \((A - \lambda I)^j\), for some integer \(j > 0\).

We consider a simple example. Let \(A\) be an \(n \times n\) real matrix having eigenvalues

\[\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > \cdots > \lambda_n > 0.\]
Assume we have an orthonormal set of vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ and a constant $\kappa \neq 0$ such that

$$Av_i = \lambda_i v_i, \quad i \neq 2,$$
$$Av_2 = \lambda_2 v_2 + \kappa v_1.$$ 

That is, $v_1$ and $v_3, \ldots, v_n$ are eigenvectors of $A$, and $v_2$ is a generalized eigenvector.

Now, we try applying the power method to $A$. Let $w \in \mathbb{R}^n$ be a nonzero vector, and consider the sequence

$$w_k = \frac{A^k w}{\|A^k w\|}.$$ 

(a) Find $\lim_{k \to \infty} w_k$. Does the power method always converge to an eigenvector of $A$?

(b) Suppose that instead $\kappa = 0$, so that $v_2$ is in fact an eigenvector of $A$ (and $A$ is diagonalizable).

Once again, find $\lim_{k \to \infty} w_k$. Does the power method always converge to an eigenvector of $A$?

Problem 3: Cholesky Factorization (40 points, pledged)

In class, we discussed the Cholesky factorization $M = LL^*$. Write MATLAB code to implement the Cholesky factorization for a Hermitian positive definite matrix $M \in \mathbb{C}^{n \times n}$. Your code should return an error if $M$ is not Hermitian, or if some $\alpha$ is nonpositive, in which case $M$ is not positive definite.

Test your code in the following way: Generate a number of random complex lower-triangular matrices $L_0$ with positive diagonal entries. For each, multiply $M = L_0 L_0^*$, and apply your Cholesky factorization algorithm to get an approximate factorization $M = LL^*$, then compute the error $\|L - L_0\|$. Report on the errors you find.