WELL-POSEDNESS FOR HYPERBOLIC PROBLEMS

We will use the familiar Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. We consider the Cauchy problem

\begin{align}
\Box u &= (\partial_t^2 - \Delta_c)u = f \in L^2((0,T) \times \Omega) \quad \text{on } [0,T] \times \Omega \\
\quad u(0) &= u_0 \in H_0^1(\Omega) \\
\quad u'(0) &= u_1 \in L^2(\Omega).
\end{align}

We will denote $\langle \cdot, \cdot \rangle$ as the inner product on $L^2(\Omega)$. If we pair the above equation with a test function $\phi \in C^1_c(0,T;H_0^1(\Omega))$, then we formally obtain

\begin{align}
\int_0^T \left[ \langle \partial_t^2 u(t), \phi(t) \rangle - \langle \Delta_c u(t), \phi(t) \rangle \right] dt &= \int_0^T \langle f(t), \phi(t) \rangle dt.
\end{align}

We say $u$ is a weak solution of (0.1) if it satisfies (0.2) for any such test functions $\phi$, $u \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, and $u$ satisfies the initial conditions in (0.1). Hence, it will be useful to define the sesquilinear form

\[ a(u,v) := \langle c\nabla u, c\nabla v \rangle, \quad u,v \in H_0^1(\Omega). \]

**Theorem 0.1.** There exists a unique weak solution $u$ to (0.1) such that $u \in L^2((0,T);H_0^1(\Omega))$ and $u' \in L^2((0,T);L^2(\Omega))$.

**Proof.** (Existence) Since $H_0^1(\Omega)$ is separable, let $\{w_j\}_{j=1}^\infty \subset H_0^1(\Omega)$ be a countable basis for $H_0^1(\Omega)$ in the following sense:

for all $m, w_1, \ldots, w_m$ are linearly independent, and,

Each $x \in H_0^1(\Omega)$ is limit of elements of the form $\sum_{\text{finite}} \xi_k w_k$, for $\xi_k \in \mathbb{C}$.

We define the approximate solution $u_m(t)$ of order $m$ of the problem in the following way

\[ u_m(t) = \sum_{k=1}^m g_{km}(t) w_k, \]

the $g_{km}$’s being determined so that

\begin{align}
\begin{align*}
a(u_m(t), w_l) + &\langle u_m''(t), w_l \rangle = \langle f(t), w_l \rangle, \quad 1 \leq l \leq m, \\
g_{km}(0) &= \xi_{km}, \\
g_{km}(0) &= \eta_{km}
\end{align*}
\end{align}
Thus, the $g_{km}$’s are determined by a second order linear ordinary differential system of the form $A\ddot{x}(t) + Cx(t) = b(t)$ for $m \times m$ constant matrices $A$, $C$ and $b(t)$ a vector containing $m$ components, which admits a unique solution since $A$ is invertible. The hope is that the sequences $\{u_m\}_{m=1}^{\infty}$ and $\{u'_m\}_{m=1}^{\infty}$ have convergent subsequences in $L^2(0,T;H^1_0(\Omega))$ and $L^2(0,T;L^2(\Omega))$ respectively that converge to a weak solution of the PDE. Using the Banach-Alaoglu theorem, it will suffice to show that these sequences are uniformly bounded which requires an energy estimate.

**Energy Estimate**

Let $u \in C^2_c([0,T], H^1_0(\Omega))$, such that $\square_u = f$ weakly in the sense of (0.3) (i.e. that $a(u(t), v) + \langle u''(t), v \rangle = \langle f(t), v \rangle, \forall v \in H^1_0(\Omega)$).

It is shown on page 213 of the textbook that $a(\cdot, \cdot)$ defines an equivalent inner product and norm on $H^1_0(\Omega)$, i.e. one has constants $A, B > 0$ such that for all $\phi \in H^1_0(\Omega)$

$$A\|\phi\|_{H^1_0} \leq a(\phi, \phi) = \langle c\nabla \phi \cdot c\nabla \phi \rangle_{L^2(\Omega)} \leq B\|\phi\|_{H^1_0}.$$  

Thus, WLOG we may interchange these two norms in all of our estimates if needed. Define the energy of $u$ at time $t$ by

$$E(u(t)) = a(u(t), u(t)) + \|u'(t)\|^2.$$  

Then using $u'(t) \in H^1_0(\Omega)$ as a test function, we have

$$\frac{d}{dt}E(u(t)) = a(u'(t), u(t)) + a(u(t), u'(t)) + \langle u''(t), u'(t) \rangle + \langle u'(t), u''(t) \rangle$$  

$$= \langle u'(t), f(t) \rangle + \langle f(t), u'(t) \rangle$$  

$$= 2\Re\langle f(t), u'(t) \rangle.$$  

Thus, we obtain

$$E(u(t)) = E(u(0)) + 2\int_0^t \Re\langle f(s), u'(s) \rangle ds.$$  

If we integrate this expression from 0 to $T$ we obtain

$$\|u\|^2_{L^2(0,T;H^1_0(\Omega))} + \|u'\|^2_{L^2(0,T;\Omega)} \leq TE(u(0)) + 2\int_0^T \int_0^t |\Re\langle f(s), u'(s) \rangle| ds dt$$  

$$\leq C \left( (E(u(0)) + \int_0^T |\langle f(s), u'(s) \rangle| ds \right)$$  

$$\leq C(E(u(0)) + \epsilon^{-1}\|f\|^2_{L^2(0,T;\Omega)} + \epsilon\|u'\|^2_{L^2(0,T;\Omega)}$$
where we use Cauchy-Schwartz and Young’s inequality in the final inequality above, and $\epsilon > 0$. With possibly a different constant $C$ and making $\epsilon < 1$ we obtain

\begin{equation}
||u||^2_{L^2((0,T);H^1_0(\Omega))} + ||u'||^2_{L^2((0,T) \times \Omega)} \leq C(E(u(0)) + ||f||^2_{L^2((0,T) \times \Omega)}).
\end{equation}

In fact, we can get an improved result showing that the map $t \to E(u(t))$ is actually Holder continuous with weaker assumptions on $u$ (see Lemma 0.2).

**Returning to the proof**

Next, by Bessel’s inequality we have

\[ ||u_m(0)||^2_{H^1_0(\Omega)} \leq ||u_0||^2_{H^1_0(\Omega)} \text{ and } ||u_m'(0)||_{L^2(\Omega)} \leq ||u_1||^2_{L^2(\Omega)} \quad \forall m. \]

If we combine these two inequalities and apply the proof of the energy inequality to $u_m$, we obtain

\[ ||u_m||^2_{L^2(0,T;H^1_0(\Omega))} + ||u_m'||^2_{L^2((0,T) \times \Omega)} \leq C(E(u(0)) + ||f||^2_{L^2((0,T) \times \Omega)}) \]

with a constant independent of $m$.

Next, apply the Banach-Alaoglu theorem to the Hilbert spaces $L^2(0,T;H^1_0(\Omega))$ and $L^2(0,T;L^2(\Omega))$ to extract subsequence $u_{m_k}$ from $u_m$ such that for some $u \in L^2(0,T;H^1_0(\Omega))$ and $v \in L^2((0,T) \times \Omega)$ we have

\[ u_{m_k} \rightharpoonup u \text{ in } L^2(0,T;H^1_0(\Omega)), \]

\[ u_{m_k}' \rightharpoonup v \text{ in } L^2((0,T) \times \Omega). \]

Now using continuity of the weak derivative, we obtain from the first convergence that $u_{m_k}' \rightharpoonup u'$ in a weak distribution sense. But by uniqueness of the weak limit, we must have $v = u'$.

Next, we check that $u$ is indeed a weak solution to our PDE. We first show that $u$ satisfies the differential equation. Take

\[ \psi = \sum_{j=1}^r \psi_j(t)w_j \]

with $\psi_j \in C^1_c((0,T))$. After multiplying equation (0.3) by $\psi_j$ and summing, we obtain for $m_k \geq r$

\[ a(u_{m_k}(t),\psi(t)) + \langle u_{m_k}''(t),\psi(t) \rangle = \langle f(t),\psi(t) \rangle \]

so after an integration by parts we get

\[ \int_0^T [a(u_{m_k}(t),\psi(t)) - \langle u_{m_k}'(t),\psi'(t) \rangle]dt = \int_0^T \langle f(t),\psi(t) \rangle dt. \]

Then letting $m_k \to \infty$ above gives

\[ \int_0^T [a(u(t),\psi(t)) - \langle u'(t),\psi'(t) \rangle]dt = \int_0^T \langle f(t),\psi(t) \rangle dt. \]

Since the $w_j$ form a basis for $H^1_0(\Omega)$, the set of such $\psi$ is actually dense in the space of functions $\psi \in L^2(0,T;H^1_0(\Omega))$ such that

\[ \psi' \in L^2((0,T) \times \Omega), \quad \psi(0) = \psi(T) = 0 \]

so $u$ indeed satisfies the differential equation.
We now show that \( u \) satisfies the first initial condition. Define the following space of functions
\[
C_T^\infty = \{ \phi | \phi \in C^\infty([0,T]), \phi(T) = \phi'(T) = 0 \},
\]
and consider the functions of the form
\[
\psi = \phi \otimes v, \quad v \in H_0^1(\Omega), \phi \in C_T^\infty([0,T]).
\]
By partial integration we have
\[
\int_0^T [\langle u'_m, \psi \rangle + \langle u_m, \psi' \rangle] dt = -\langle u_m(0), \psi(0) \rangle.
\]
The right hand side converges to \( \langle u_0, \psi(0) \rangle \) if \( k \to \infty \). The left hand side converges to
\[
\int_0^T [\langle u', \psi \rangle + \langle u, \psi' \rangle] dt = \langle u(0), \psi(0) \rangle.
\]
Hence
\[
\langle u(0), \psi(0) \rangle = \langle u_0, \psi(0) \rangle.
\]
It follows that \( u \) satisfies the first initial condition.

Using the equation (0.3) for \( u_{m_k} \) tested with \( \psi \), one has an extra term due to the partial integration:
\[
\int_0^T [a(u_{m_k}, \psi) - \langle u'_{m_k}, \psi' \rangle] dt = \int_0^T (f, \psi) dt + \langle u_{m_k}'(0), \psi(0) \rangle.
\]
Letting \( k \to \infty \), we obtain
\[
\int_0^T [a(u(t), \psi(t)) - \langle u'(t), \psi'(t) \rangle] dt = \int_0^T (f(t), \psi(t)) dt + \langle u_1, \psi(0) \rangle.
\]
Now let \( \phi_k \in C_\infty^\infty((-,T)) \) such that
\[
\phi_k \to H(-t) \text{ pointwise,}
\]
\( \phi_k(0) = 1 \) for each \( k \), and \( \phi'_k \to -\delta(0) \) in distribution (\( H(t) \) here denotes the Heaviside function). So replacing \( \psi \) above with \( \psi_k = \phi_k \otimes v \) instead and letting \( k \to \infty \) means the first term on the left goes to 0 as well as the first term on the right (since the support of \( \phi_k \) shrinks to 0 inside \((0,T)\)). But \( -\int_0^T \langle u'(t), \psi'_k(t) \rangle dt \to \langle u'(0), v \rangle \). We thus obtain
\[
\langle u'(0), v \rangle = \langle u_1, v \rangle \quad \forall v \in L^2(\Omega).
\]
Hence, \( u'(0) = u_1 \) as desired.

**Uniqueness**

Suppose \( u_1 \) and \( u_2 \) are two weak solutions of the PDE (0.1). Then \( u = u_1 - u_2 \) solves \( \Box u = 0 \), \( u(0) = 0 \) and \( u'(0) = 0 \). If we knew that \( u'' \in L^2(0,T;L^2(\Omega)) \), then our energy estimate (0.4) goes through as before to show \( u \equiv 0 \). Since we do not have this type of regularity in time, we proceed to regularize in time.
Thus, let \( \rho_m, m = 1, 2, \ldots \) be a regularizing sequence, that is
\[
\rho_m(t) \geq 0 \\
\int \rho_m(t) dt = 1 \\
\text{supp } \rho_m \to \{0\}.
\]
Let \( \epsilon > 0 \). There is an \( \mathcal{M} \) such that \( \text{supp } \rho_m \subset (-\epsilon, \epsilon) \) for all \( m > \mathcal{M} \).

Set \( \phi(t) = \rho_m(s-t)(\rho_m \ast u)(s) \), then, for \( s \in [\epsilon, T - \epsilon] \), we have that \( \phi \in C^\infty_c([0, T], H^1_0(\Omega)) \). Thus, \( u \) satisfies (0.2) with this choice of \( v \) and we denote \( u_m(t) = \rho_m \ast u(t) \). Notice that \( \int_0^T \rho_m(s-t)u(t) dt = \rho_m \ast u(s) \), so using (0.2) with \( u \) paired with \( \phi \) (\( f \) being 0 in (0.2)) gives
\[
(0.5) \quad a(u_m(s), u_m'(s)) + \langle u_m''(s), u_m'(s) \rangle = \langle \rho_m \ast 0, u_m'(s) \rangle = 0.
\]
Thus, the energy estimate proceeds exactly as before to show
\[
\int_0^T E(u_m(t)) dt \leq C \int_0^T E(u_m(0)).
\]
The mollification we have constructed ensures that
\[
\begin{align*}
 u_m & \to u \quad \text{in } L^2(0, T; H^1_0(\Omega)) \\
 u_m' & \to u' \quad \text{in } L^2(0, T; L^2(\Omega)),
\end{align*}
\]
Then letting \( m \to \infty \) and using \( E(u(0)) = 0 \) shows
\[
\int_0^T E(u(t)) dt = 0
\]
so that \( u \equiv 0 \). \( \square \)

0.1. Hölder-1/2 continuity of the energy function.

**Lemma 0.2.** Let \( u \in L^2(0, T; H^1_0(\Omega)), u' \in L^2(0, T; L^2(\Omega)), f \in L^2(0, T; L^2(\Omega)) \). If \( u \) satisfies (0.2), then the function \( t \to E(u(t)) \) is Hölder continuous of order 1/2 on a set \( I_0 \) that differs from \((0, T)\) by a set of measure 0.

**Proof.** Using mollifiers, the proof of (0.5) shows with the notation in that section that
\[
a(u_m(s), u_m'(s)) + \langle u_m''(s), u_m'(s) \rangle = \langle f_m(s), u_m'(s) \rangle
\]
and hence
\[
\frac{d}{dt} E(u_m(t)) = 2\Re \langle f_m(t), u_m'(t) \rangle.
\]
Looking at the right hand side shows the limit as \( m \to \infty \) actually exists in \( L^1((0, T)) \) and we have
\[
\frac{d}{dt} E(u(t)) = 2\Re \langle f(t), u'(t) \rangle
\]
in \( L^1((0, T)) \). It follows that \( \frac{dE(u(t))}{dt} \) is in \( L^1((0, T)) \) so that \( E(u(t)) \) is bounded in \([0, T]\); hence, \( \|u(t)\|_{H^1_0(\Omega)}, \|u'(t)\|_{L^2(\Omega)} \) are bounded as well. Another consequence
is $E(u(t))$ is absolutely continuous on a set $I_0$ that differs from $[0, T]$ by a set of measure zero. Integrating this expression from $s$ to $t$, $s < t \in I_0$, gives

\[
|E(u(t)) - E(u(s))| = 2 \left| \int_s^t \Re \langle f(r), u'(r) \rangle dr \right| \leq 2 \left( \int_s^t 1 dr \right)^{1/2} \left( \int_s^t |\langle f(r), u'(r) \rangle|^2 dr \right)^{1/2}
\]

\[
\leq 2 \sqrt{t - s} \left( \int_0^T |\langle f(r), u'(r) \rangle|^2 dr \right)^{1/2}
\]

\[
\leq 2C \sqrt{t - s} ||f||_{L^2(0, T; L^2(\Omega))}
\]

where $C > 0$ is a uniform constant such that $||u'(t)||_{L^2(\Omega)} \leq C$ for $t \in [0, T]$. 