WELL-POSEDNESS FOR HYPERBOLIC PROBLEMS

We will use the familiar Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. We consider the Cauchy problem

$$\begin{aligned} \square_c u = (\partial_t^2 - \Delta_c)u = f \in L^2((0,T)\times\Omega) \text{ on } [0,T]\times\Omega \\ u(0) = u_0 \in H_0^1(\Omega) \\ u'(0) = u_1 \in L^2(\Omega). \end{aligned} \tag{0.1}$$

We will denote $\langle\cdot,\cdot\rangle$ as the inner product on $L^2(\Omega)$. If we pair the above equation with a test function $\phi \in C_c^1(0,T;H_0^1(\Omega))$, then we formally obtain

$$\int_0^T [\langle \partial_t^2 u(t), \phi(t)\rangle - \langle \Delta_c u(t), \phi(t)\rangle] dt = \int_0^T \langle f(t), \phi(t)\rangle dt.$$

$$\Rightarrow \int_0^T [\langle c^2 \nabla u(t), \nabla\phi\rangle - \langle \partial_t u(t), \partial_t \phi(t)\rangle] dt = \int_0^T \langle f(t), \phi(t)\rangle dt \tag{0.2}$$

using an application of Green's theorem and integration by parts where any integral over a subset of $\partial([0,T]\times\Omega)$ vanishes due to support assumptions on ϕ.

We say u is a *weak solution* of (0.1) if it satisfies (0.2) for any such test functions ϕ, $u \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, and u satisfies the initial conditions in (0.1). Hence, it will be useful to define the sesquilinear form

$$a(u,v) := \langle c\nabla u, c\nabla v\rangle, \quad u, v \in H_0^1(\Omega).$$

Theorem 0.1. *There exists a unique weak solution u to (0.1) such that $u \in L^2((0,T);H_0^1(\Omega))$ and $u' \in L^2((0,T);L^2(\Omega))$.*

Proof. (Existence) Since $H_0^1(\Omega)$ is separable, let $\{w_j\}_{j=1}^\infty \subset H_0^1(\Omega)$ be a countable basis for $H_0^1(\Omega)$ in the following sense:

for all $m, w_1, \ldots, w_m$ are linearly independent, and,

Each $x \in H_0^1(\Omega)$ is limit of elements of the form $\sum_{\text{finite}} \xi_k w_k$, for $\xi_k \in \mathbb{C}$.

We define the approximate solution $u_m(t)$ of order m of the problem in the following way

$$u_m(t) = \sum_{k=1}^m g_{km}(t) w_k,$$

the g_{km}'s being determined so that

$$\begin{aligned} a(u_m(t), w_l) + \langle u_m''(t), w_l\rangle = \langle f(t), w_l\rangle, \quad 1 \le l \le m, \\ g_{km}(0) = \xi_{km}, \qquad g_{km}'(0) = \eta_{km} \end{aligned} \tag{0.3}$$

with

$$\sum_{k=1}^{m} \xi_{km} w_k \to u_0 \quad \text{in } H_0^1(\Omega) \text{ as } m \to \infty,$$
$$\sum_{k=1}^{m} \eta_{km} w_k \to u_1 \quad \text{in } L^2(\Omega) \text{ as } m \to \infty.$$

Thus, the g_{km}'s are determined by a second order linear ordinary differential system of the form $A\ddot{x}(t)+Cx(t) = b(t)$ for $m \times m$ constant matrices A, C and $b(t)$ a vector containing m components, which admits a unique solution since A is invertible. The hope is that the sequences $\{u_m\}_{m=1}^{\infty}$ and $\{u_m'\}_{m=1}^{\infty}$ have convergent subsequences in $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$ respectively that converge to a weak solution of the PDE. Using the Banach-Alaoglu theorem, it will suffice to show that these sequences are uniformly bounded which requires an energy estimate.

Energy Estimate

Let $u \in C_c^2([0,T], H_0^1(\Omega))$, such that $\Box_c u = f$ weakly in the sense of (0.3) (i.e. that $a(u(t),v) + \langle u''(t), v\rangle = \langle f(t), v\rangle, \forall v \in H_0^1(\Omega)$).

It is shown on page 213 of the textbook that $a(\cdot,\cdot)$ defines an equivalent inner product and norm on $H_0^1(\Omega)$, i.e. one has constants $A, B > 0$ such that for all $\phi \in H_0^1(\Omega)$

$$A||\phi||_{H_0^1} \leq a(\phi,\phi) = \langle c\nabla\phi \cdot c\nabla\phi\rangle_{L^2(\Omega)} \leq B||\phi||_{H_0^1(\Omega)}.$$

Thus, WLOG we may interchange these two norms in all of our estimates if needed. Define the energy of u at time t by

$$E(u(t)) = a(u(t),u(t)) + ||u'(t)||^2.$$

Then using $u'(t) \in H_0^1(\Omega)$ as a test function, we have

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= a(u'(t),u(t)) + a(u(t),u'(t)) + \langle u''(t), u'(t)\rangle + \langle u'(t), u''(t)\rangle \\ &= \langle u'(t), f(t)\rangle + \langle f(t), u'(t)\rangle \\ &= 2\Re\langle f(t), u'(t)\rangle.\end{aligned}$$

Thus, we obtain

$$E(u(t)) = E(u(0)) + 2\int_0^t \Re\langle f(s), u'(s)\rangle ds.$$

If we integrate this expression from 0 to T we obtain

$$\begin{aligned}||u||^2_{L^2(0,T;H_0^1(\Omega))} + ||u'||^2_{L^2((0,T)\times\Omega)} &\leq TE(u(0)) + 2\int_0^T\int_0^t |\Re\langle f(s), u'(s)\rangle| dsdt \\ &\leq C\left((E(u(0)) + \int_0^T |\langle f(s), u'(s)\rangle| ds\right) \\ &\leq C(E(u(0)) + \epsilon^{-1}||f||^2_{L^2(0,T;\Omega)}) + \epsilon||u'||^2_{L^2((0,T)\times\Omega)}\end{aligned}$$

where we use Cauchy-Schwartz and Young's inequality in the final inequality above, and $\epsilon > 0$. With possibly a different constant C and making $\epsilon < 1$ we obtain

$$||u||^2_{L^2(0,T;H^1_0(\Omega))} + ||u'||^2_{L^2((0,T)\times\Omega)} \leq C(E(u(0)) + ||f||^2_{L^2((0,T)\times\Omega)}). \tag{0.4}$$

In fact, we can get an improved result showing that the map $t \to E(u(t))$ is actually Holder continuous with weaker assumptions on u (see Lemma 0.2).

Returning to the proof

Next, by Bessel's inequality we have

$$||u_m(0)||^2_{H^1_0(\Omega)} \leq ||u_0||^2_{H^1_0(\Omega)} \text{ and } ||u'_m(0)||_{L^2(\Omega)} \leq ||u_1||^2_{L^2(\Omega)} \ \forall m.$$

If we combine these two inequalities and apply the proof of the energy inequality to u_m, we obtain

$$||u_m||^2_{L^2(0,T;H^1_0(\Omega))} + ||u'_m||^2_{L^2((0,T)\times\Omega)} \leq C(E(u(0)) + ||f||^2_{L^2(0,T;\Omega)})$$

with a constant independent of m.

Next, apply the Banach-Alaoglu theorem to the Hilbert spaces $L^2(0,T;H^1_0(\Omega))$ and $L^2(0,T;L^2(\Omega))$ to extract subsequence u_{m_k} from u_m such that for some $u \in L^2(0,T;H^1_0(\Omega))$ and $v \in L^2((0,T)\times\Omega)$ we have

$$u_{m_k} \rightharpoonup u \text{ in } L^2(0,T;H^1_0(\Omega))$$
$$u'_{m_k} \rightharpoonup v \text{ in } L^2((0,T)\times\Omega).$$

Now using continuity of the weak derivative, we obtain from the first convergence that $u'_{m_k} \rightharpoonup u'$ in a weak distribution sense. But by uniqueness of the weak limit, we must have $v = u'$.

Next, we check that u is indeed a weak solution to our PDE. We first show that u satisfies the differential equation. Take

$$\psi = \sum_{j=1}^{r} \psi_j(t) w_j$$

with $\psi_j \in C^1_c((0,T))$. After multiplying equation (0.3) by ψ_j and summing, we obtain for $m_k \geq r$

$$a(u_{m_k}(t), \psi(t)) + \langle u''_{m_k}(t), \psi(t)\rangle = \langle f(t), \psi(t)\rangle$$

so after an integration by parts we get

$$\int_0^T [a(u_{m_k}(t), \psi(t)) - \langle u'_{m_k}(t), \psi'(t)\rangle] dt = \int_0^T \langle f(t), \psi(t)\rangle dt.$$

Then letting $m_k \to \infty$ above gives

$$\int_0^T [a(u(t), \psi(t)) - \langle u'(t), \psi'(t)\rangle] dt = \int_0^T \langle f(t), \psi(t)\rangle dt.$$

Since the w_j form a basis for $H^1_0(\Omega)$, the set of such ψ is actually dense in the space of functions $\psi \in L^2(0,T;H^1_0(\Omega))$ such that

$$\psi' \in L^2((0,T)\times\Omega), \qquad \psi(0) = \psi(T) = 0$$

so u indeed satisfies the differential equation.

We now show that u satisfies the first initial condition. Define the following space of functions

$$C_T^\infty = \{\phi | \phi \in C^\infty([0,T]), \phi(T) = \phi'(T) = 0\},$$

and consider the functions of the form

$$\psi = \phi \otimes v, \qquad v \in H_0^1(\Omega), \phi \in C_T^\infty([0,T]).$$

By partial integration we have

$$\int_0^T [\langle u'_{m_k}, \psi\rangle + \langle u_{m_k}, \psi'\rangle] dt = -\langle u_{m_k}(0), \psi(0)\rangle.$$

The right hand side converges to $\langle u_0, \psi(0)\rangle$ if $k \to \infty$. The left hand side converges to

$$\int_0^T [\langle u', \psi\rangle + \langle u, \psi'\rangle] dt = \langle u(0), \psi(0)\rangle.$$

Hence

$$\langle u(0), \psi(0)\rangle = \langle u_0, \psi(0)\rangle.$$

It follows that u satisfies the first initial condition.

Using the equation (0.3) for u_{m_k} tested with ψ, one has an extra term due to the partial integration:

$$\int_0^T [a(u_{m_k}, \psi) - \langle u'_{m_k}, \psi'\rangle] dt = \int_0^T \langle f, \psi\rangle dt + \langle u'_{m_k}(0), \psi(0)\rangle.$$

Letting $k \to \infty$, we obtain

$$\int_0^T [a(u(t), \psi(t)) - \langle u'(t), \psi'(t)\rangle] dt = \int_0^T \langle f(t), \psi(t)\rangle dt + \langle u_1, \psi(0)\rangle.$$

Now let $\phi_k \in C_c^\infty((-\infty, T])$ such that

$$\phi_k \to H(-t) \text{ pointwise},$$

$\phi_k(0) = 1$ for each k, and $\phi'_k \to -\delta(0)$ in distribution ($H(t)$ here denotes the Heaviside function). So replacing ψ above with $\psi_k = \phi_k \otimes v$ instead and letting $k \to \infty$ means the first term on the left goes to 0 as well as the first term on the right (since the support of ϕ_k shrinks to $\emptyset$ inside $(0,T)$). But $-\int_0^T \langle u'(t), \psi'_k(t)\rangle dt \to \langle u'(0), v\rangle$. We thus obtain

$$\langle u'(0), v\rangle = \langle u_1, v\rangle \qquad \forall v \in L^2(\Omega).$$

Hence, $u'(0) = u_1$ as desired.

Uniqueness

Suppose u_1 and u_2 are two weak solutions of the PDE (0.1). Then $u = u_1 - u_2$ solves $\Box_c u = 0$, $u(0) = 0$ and $u'(0) = 0$. If we knew that $u'' \in L^2(0,T; L^2(\Omega))$, then our energy estimate (0.4) goes through as before to show $u \equiv 0$. Since we do not have this type of regularity in time, we proceed to regularize in time.

Thus, let ρ_m, $m = 1, 2, \dots$ be a regularizing sequence, that is

$$\rho_m(t) \geq 0$$
$$\int \rho_m(t)dt = 1$$
$$\text{supp}\rho_m \to \{0\}.$$

Let $\epsilon > 0$. There is an M such that $\text{supp}\rho_m \subset (-\epsilon, \epsilon)$ for all $m > M$.

Set $\phi(t) = \rho_m(s-t)(\rho_m' \star u)(s)$, then, for $s \in [\epsilon, T-\epsilon]$, we have that $\phi \in C_c^\infty([0,T], H_0^1(\Omega))$. Thus, u satisfies (0.2) with this choice of v and we denote $u_m(t) = \rho_m \star u(t)$. Notice that $\int_0^T \rho_m(s-t)u(t)dt = \rho_m \star u(s)$, so using (0.2) with u paired with ϕ (f being 0 in (0.2)) gives

$$a(u_m(s), u_m'(s)) + \langle u_m''(s), u_m'(s)\rangle = \langle \rho_m \star 0, u_m'(s)\rangle = 0. \tag{0.5}$$

Thus, the energy estimate proceeds exactly as before to show

$$\int_0^T E(u_m(t))dt \leq C \int_0^T E(u_m(0)).$$

The mollification we have constructed ensures that

$$u_m \to u \qquad \text{in } L^2(0,T;H_0^1(\Omega))$$
$$u_m' \to u' \qquad \text{in } L^2(0,T;L^2(\Omega)).$$

Then letting $m \to \infty$ and using $E(u(0)) = 0$ shows

$$\int_0^T E(u(t))dt = 0$$

so that $u \equiv 0$. □

0.1. **Hölder-1/2 continuity of the energy function.**

Lemma 0.2. *Let $u \in L^2(0,T;H_0^1(\Omega))$, $u' \in L^2(0,T;L^2(\Omega))$, $f \in L^2(0,T;L^2(\Omega))$. If u satisfies (0.2), then the function $t \to E(u(t))$ is Hölder continuous of order $1/2$ on a set I_0 that differs from $(0,T)$ by a set of measure 0.*

Proof. Using mollifiers, the proof of (0.5) shows with the notation in that section that

$$a(u_m(s), u_m'(s)) + \langle u_m''(s), u_m'(s)\rangle = \langle f_m(s), u_m'(s)\rangle$$

and hence

$$\frac{d}{dt}E(u_m(t)) = 2\Re\langle f_m(t), u_m'(t)\rangle.$$

Looking at the right hand side shows the limit as $m \to \infty$ actually exists in $L^1((0,T))$ and we have

$$\frac{d}{dt}E(u(t)) = 2\Re\langle f(t), u'(t)\rangle$$

in $L^1((0,T))$. It follows that $\frac{dE(u(t))}{dt}$ is in $L^1((0,T))$ so that $E(u(t))$ is bounded in $[0,T]$; hence, $||u(t)||_{H_0^1(\Omega)}, ||u'(t)||_{L^2(\Omega)}$ are bounded as well. Another consequence is $E(u(t))$ is absolutely continuous on a set I_0 that differs from $[0,T]$ by a set of measure zero. Integrating this expression from s to t, $s < t \in I_0$, gives

$$\begin{aligned}
|E(u(t)) - E(u(s))| = 2\left|\int_s^t \Re\langle f(r), u'(r)\rangle dr\right| &\leq 2\left(\int_s^t 1dr\right)^{1/2}\left(\int_s^t |\langle f(r), u'(r)\rangle|^2 dr\right)^{1/2} \\
&\leq 2\sqrt{t-s}\left(\int_0^T |\langle f(r), u'(r)\rangle|^2 dr\right)^{1/2} \\
&\leq 2C\sqrt{t-s}||f||_{L^2(0,T;L^2(\Omega))}
\end{aligned}$$

where $C > 0$ is a uniform constant such that $||u'(t)||_{L^2(\Omega)} \leq C$ for $t \in [0,T]$.