MATH/CAAM 423: Partial Differential Equations

Problem Set 8: Galerkin Well-Posedness and Variational Problems
Due: Wednesday, November 23, 2016

Problem 1: Rayleigh Quotients (30 points)

Do problems 18.1 and 18.2 from the book.

Problem 2: Acoustic Wave Equation (40 points, pledged)

In class, we showed existence and uniqueness of solutions to a wave equation problem using a Galerkin method. In this problem, consider the slightly different acoustic wave equation initial-boundary value problem

$$\begin{cases} \partial_t^2 u = c^2(x)\Delta u, \\ u\big|_{t=0} = f(x), \\ \partial_t u\big|_{t=0} = g(x), \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

As usual, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and c is bounded and bounded away from zero: $0 < c_{\min} \leq c \leq c_{\max} < \infty$ for constants $c_{\min}, c_{\max}$.

By modifying the proof from class, *carefully* prove existence and uniqueness of solutions $u \in L^2((0,T); H_0^1(\Omega))$ given data $f \in H_0^1(\Omega)$, $g \in L^2(\Omega)$.

Problem 3: More Galerkin (30 points)

(a) The equation $\partial_t^2 u + c(x)\nabla(c(x)\nabla u) = 0$ is elliptic, so in general $u\big|_{t=0}$ and $\partial_t u\big|_{t=0}$ cannot both be specified. What goes wrong when adapting the Galerkin proof of existence and uniqueness to this equation?

(b) Let $\Omega = (0,\pi) \subset \mathbb{R}$, and let $\{v_n\}_{n=1}^{\infty}$, $v_n(x) = \sin(nx)$ be an orthogonal basis for $H_0^1(\Omega)$. Consider the wave PDE $\partial_t^2 u - c(x)\nabla(c(x)\nabla u) = 0$ with wave speed $c(x) = \sqrt{1+\cos x}$, satisfying Dirichlet boundary conditions and initial conditions $u(0,x) = u_0 \in H_0^1(\Omega)$, $\partial_t u(0,x) = u_1 \in L^2(\Omega)$.

Write down explicitly the matrix ODE satisfied by the coefficients $g_{km}(t)$ of the m^{th} Galerkin approximate solution $u_m(x,t) = \sum_{k=1}^{m} v_k(x) g_{km}(t)$.