MATH/CAAM 423: Partial Differential Equations

Problem Set 7: Inner Product Spaces and Solvability
Due: Tuesday, November 8, 2016

Problem 1: Sobolev Spaces (60 points)

When first studying PDEs, it seems natural to use our familiar function spaces $C^k(X)$ of k times differentiable functions on a domain X. For many purposes though, it's much better to consider *Sobolev spaces* $W^{s,p}$ instead. Roughly, these are spaces of functions which have s derivatives all belonging to L^p, for $1 \leq p \leq \infty$ (but watch out, s might not be an integer, and it might not be positive!)

In this problem, we will look at the common case $p = 2$, where the space is traditionally called H^s, on Euclidean space $\mathbb{R}^n$. Now, $H^s(\mathbb{R}^n)$ can be defined in several ways; we will use the Fourier transform. For $s \geq 0$, define the H^s inner product

$$\langle f, g \rangle_{H^s} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}, \qquad \text{for } f, g \in L^2(\mathbb{R}^n)$$

(which may or not converge for given $f, g \in L^2(\mathbb{R}^n)$). Define $H^s(\mathbb{R}^n)$ as

$$H^s(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \|f\|_{H^s} < \infty\} \qquad \text{with } \|f\|_{H^s} = \sqrt{\langle f, f \rangle_{H^s}}.$$

That is, $H^s(\mathbb{R}^n)$ consists of all functions whose H^s norm is finite. Note that the H^0 inner product is the same as the L^2 inner product, so $H^0(\mathbb{R}^n)$ is the same as $L^2(\mathbb{R}^n)$

(a) Show, for s an integer, that $f \in H^s$ if and only if all the distributional derivatives of f up to order s are equal to L^2 functions. For simplicity, you may assume $n = 1$.

(b) Prove that $H^s(\mathbb{R}^n)$ is a Hilbert space w.r.t. the H^s norm. You can use the fact that $L^2(\mathbb{R}^n)$ is complete.

(c) Show that $H^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ in the H^s norm.

(d) For this part only, we replace the H^s norm with the L^2 norm. The dual space of $H^s(\mathbb{R}^n)$ with respect to the L^2 norm lies in the space of distributions $\mathcal{D}'(\mathbb{R}^n)$, since $C_c^\infty(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$. Characterize this dual space in terms of its Fourier transform.

(e) What is the dual space of $H^s(\mathbb{R}^n)$ with respect to the H^s norm?

Problem 2: Well-Posedness (40 points, pledged)

Do problem 17.8 in the textbook.