MATH/CAAM 423: Partial Differential Equations

Problem Set 3: Second-order Linear PDE
Due: Thursday, September 29, 2016

Problem 1: Finite Speed of Propagation (30 points)

Do problem 7.3 in the textbook on finite speed of propagation and uniqueness for the variable-speed wave equation.

Problem 2: Standing Waves (20 points)

The wave equation has solutions that are periodic in time. These model vibrations, or *standing waves*, in a medium. A simple vibrational mode $u(x,t)$ ($x \in \mathbb{R}^n$) can be represented in the form

$$u(x,t) = v(x)e^{i\omega t}$$

for some angular frequency ω. Substituting such a u into the (constant speed) wave equation $\partial_t^2 u = c^2 \Delta u$, we find that $u(x,t)$ solves the wave equation if and only if $v(x)$ is a solution of

$$\Delta v = -\omega^2 v.$$

This new PDE is known as the *Helmholtz equation.* By examining its second-order terms on the left, we can see the Helmholtz equation is elliptic.

(a) Suppose the domain of x is $[0,1] \subset \mathbb{R}$, and Dirichlet boundary conditions $v(0) = v(1) = 0$ are given. Does the Helmholtz equation have a unique solution? (Your answer will depend on ω.) What does this say about the existence of standing waves with these boundary conditions?

(b) Let the domain of x be $[0,1] \subset \mathbb{R}$ as before, with Dirichlet boundary conditions $v(0) = a$, $v(1) = b$. Under what conditions on a, b, ω does the Helmholtz equation have a solution?

Problem 3: Distributional Waves (25 points)

Prove that any distributional solution $u(x,t) \in \mathcal{D}'(\mathbb{R}^2)$ to the 1D constant coefficient wave equation $(\partial_t^2 - c^2\partial_x^2)u = 0$ has the form

$$u(x,t) = f(x+ct) + g(x-ct)$$

for some distributions $f, g \in \mathcal{D}'(\mathbb{R})$. Conversely, prove any u of this form is a distributional solution of the wave equation.

Here, "$f(x+ct)$" is the distribution on $\mathbb{R}^2$ defined by

$$\langle f(x+ct), \phi \rangle = \int_{\mathbb{R}} \langle f, \phi_{(t)} \rangle \, dt, \qquad \phi_{(t)}(x) = \phi(x-ct, t).$$

for any $\phi(x,t) \in C_c^\infty(\mathbb{R}^2)$. This is equivalent to our usual interpretation of $f(x+ct)$ for functions, because if f is actually a function, then by the change of variables $x' = x + ct$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x+ct)\phi(x,t)\, dx\, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x')\phi(x'-ct,t)\, dx'\, dt = \int_{\mathbb{R}} \langle f, \phi_{(t)} \rangle \, dt.$$

Problem 4: More Interesting PDE (25 points, pledged)

Consider the second-order linear PDE with boundary conditions on the x- and y-axes:

$$\begin{cases} \left(x\dfrac{\partial^2}{\partial x^2} + y\dfrac{\partial^2}{\partial y^2} + 2\operatorname{sgn} x \dfrac{\partial}{\partial x} + 2\operatorname{sgn} y \dfrac{\partial}{\partial y} \right) u = 0, & (x, y \neq 0) \\ u(0, y) = -y, & \\ u(x, 0) = +x. & \end{cases}$$

Recall that $\operatorname{sgn} x$ is the signum function, defined by

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Find a solution $u \in C^0(\mathbb{R}^2)$ of this problem. Is u unique?