MATH/CAAM 423: Partial Differential Equations

Problem Set 1
Due: September 9, 2016

Problem 1: Semilinear PDE (30 points)

Consider the semilinear, first-order PDE on $\mathbb{R}^2$:

$$-yu_x + xu_y = f(x, y, u),$$

with a right-hand side f to be specified later.

(a) Find $u(x, y)$ if $f = 0$ and the initial conditions are $u(x, 0) = x^4 - x^2$ for $x \geq 0$.

(b) If $f(x, y, u) > 0$ for all x, y, u, prove that the PDE has no global solution (i.e., a solution u defined everywhere).

(c) Find a (nonzero) choice for f for which the PDE *does* have a global solution (that is not constant along characteristics).

Problem 2: More Burgers' (10 points)

Recall from the textbook Burgers' equation with an initial condition ϕ:

$$\begin{cases} u_t + uu_x = 0, \\ u(0, x) = \phi(x). \end{cases}$$

Sketch or plot the characteristics in the (t, x)-plane for the initial condition $\phi(x) = e^{-x^2}$.

Problem 3: Convergence to a Minimizer (35 points)

Consider the Euler–Lagrange problem for the 1D Poisson equation on an interval $I = [a, b]$ with boundary conditions. Let

$$\mathcal{U} = \{u \in C^1(I) : u(a) = A,\, u(b) = B\},$$

with A, B constants, and define the infimum energy

$$E_0 = \inf_{u \in \mathcal{U}} E(u), \qquad E(u) = \int_I |\nabla u|^2 \, dx.$$

(a) Prove that any sequence $\{u_k\} \subset \mathcal{U}$ with $E(u_k) \to E_0$ must converge uniformly to the unique $u_0 \in \mathcal{U}$ minimizing E.

Hint: you may want to start with the case $A = B$.

(b) Suppose we change the energy functional to $E(u) = \int_I |\nabla u|^{1/2} \, dx$. The situation is the opposite! Prove there is no minimizer in this case, if $A \neq B$. Furthermore, find a sequence $\{u_k\} \subset \mathcal{U}$ with $E(u_k) \to E_0$ that has no pointwise convergent subsequences.

Problem 4: Elastic Stresses à la Euler–Lagrange (25 points)

In this problem, you will derive an equation for how an elastic object stretches when its surface is deformed (say by an outside force). Its deformation can be described by a displacement vector field $\mathbf{u}$— so for each (x, y), $\mathbf{u}(x, y)$ is a vector describing how much the material originally at (x, y) has been moved. Let's write $\mathbf{u}(x, y)$ in components as $(u(x, y), v(x, y))$.

In response to a surface deformation, an elastic object tries to minimize *strain energy.* Strain is a 2×2 symmetric matrix S

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix},$$

defined by

$$S = \frac{1}{2}(D\mathbf{u} + (D\mathbf{u})^T),$$

where $D\mathbf{u}$ is the total derivative of $\mathbf{u}$:

$$D\mathbf{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

For our object, the strain energy F is defined in terms of strain by

$$F(u_x, u_y, v_x, v_y) = S_{11}^2 + S_{12}^2 + S_{12}^2 + S_{22}^2.$$

Our ultimate goal would be to find the functions $u(x, y)$, $v(x, y)$ that minimize the total strain energy, $\iint F(u_x, u_y, v_x, v_y)\, dx\, dy$.[1] This is an Euler–Lagrange equation, but since it has two unknowns the formula derived in class does not apply.

By mimicking the variation method from class for solving Euler–Lagrange equations, find two PDEs that u and v must satisfy in order to minimize total strain energy.

[1] Typically, u and v will be known at the object's boundary, and we must find them in its interior.