Seismic inverse scattering in the downward continuation approach

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Abstract

Seismic data are commonly modeled by a linearization around a smooth background medium in combination with a high frequency approximation. The perturbation of the medium coefficient is assumed to contain the discontinuities. This leads to two inverse problems, first the linearized inverse problem for the perturbation, and second the estimation of the background, which is a priori unknown (velocity estimation). Here we give a reconstruction formula for the linearized problem using the downward continuation approach. The reconstruction is done microlocally, up to an explicitly given pseudodifferential factor that depends on the aperture. Our main result is a characterization of the wave-equation angle transform, derived from downward continuation, that generates the common image point gathers as an invertible Fourier integral operator, microlocally. We show that the common image point gathers obtained with this particular angle transform are free of so called kinematic artifacts, even in the presence of caustics. The assumption is that the rays in the background that are associated with the reflections due to the medium perturbation are nowhere horizontal. Finally, making use of the mentioned angle transform, pseudodifferential annihilators of the data are constructed. These annihilators detect whether the data are contained in the range of the modeling operator, which is the precise criterion in migration velocity analysis to determine whether a background medium is acceptable, even in the presence of caustics.

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1. Introduction

In reflection seismology one places point sources and point receivers on the earth’s surface. Each source generates acoustic waves in the subsurface, that are reflected where the medium properties vary discontinuously. The recorded reflections that can be observed in the data are used to reconstruct these discontinuities. In this

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paper, together with an earlier paper, which will be referred to as Paper I [27], we develop an inverse scattering method using a downward continuation approach. The notion of downward continuation refers to the continuation of surface seismic data into the subsurface. The purpose is to provide a mathematical foundation of the downward continuation approach to seismic data processing. (In particular that part of seismic data processing that involves migration.)

Our main analytical tools will be from microlocal analysis, see e.g. the general references [9,32,12–14]. This should not come as a surprise. Indeed, the successful formation of images by migration is based on the propagation of wave fronts along the rays of geometrical acoustics. The theory of microlocal analysis precisely captures this behavior of the solutions of wave equations, and is therefore a natural tool for the analysis of seismic migration. A number of the references below also use microlocal techniques.

We study the inversion of seismic data with an acoustic model. Let $z \in \mathbb{R}$ denote the vertical (depth) coordinate, $x \in \mathbb{R}^{n-1}$ the horizontal coordinate(s) and $t$ the time, and let $c = c(z, x)$ be the wave speed, then, assuming constant density of mass, the scalar wave equation for acoustics is given by

$$Pu = f, \quad P = c(z, x)^{-2} \partial_{x}^2 - \partial_{z}^2 - \sum_{j=1}^{n-1} \partial_{x_j}^2,$$  \hspace{1cm} (1)$$

When the wavespeed $c$ is smooth, the singularities of the solution propagate along curved trajectories, characteristics, while discontinuities in the wavespeed lead to reflections.

In practice, to model seismic reflection data (1) is not used directly, but a linearization is invoked, together with a separation of “scales”. The linearization is in the coefficient $c$ around a smooth background $c_0$, $c = c_0 + \delta c$. We denote the Green’s function of (1) with velocity $c_0$ by $G$, with $G = 0$ for $t < 0$. The perturbation $\delta c$ is assumed to contain the short wavelength variations of $c$, and has no long wavelength components. Note that this is only a partial linearization: The dependence of the wavefield on $c_0$ is still nonlinear. The high-frequency part of the perturbation in the Green’s function, which we denote by $\delta G$, models the once reflected waves, whence this approximation is called the high-frequency single scattering approximation.

In the seismic inverse problem, both the smooth background $c_0$ and the perturbation $\delta c$ are unknown and have to be reconstructed from the reflection data. This leads to two related inverse problems. The first one is the linear inverse problem of reconstructing $\delta c$ given $c_0$. The solution of this problem is used in the estimation of $c_0$, the second problem, which is highly nonlinear. The purpose of this paper is to develop and analyze methods based on downward continuation for these two inverse problems; the advantages of the downward continuation approach in the presence of caustics are elucidated in Section 1.4 below.

1.1. Microlocal analysis and linearization of the scattering problem

The bicharacteristics, $\eta$ with components $(\eta_z, \eta_x, \eta_c, \eta_{\xi})$ in phase space, that describe the propagation of singularities by the wave equation are the solutions to a Hamilton system, derived from the principal symbol of
P, and will be parameterized by initial position \((z_0, x_0)\), take-off direction \(x \in S^{n-1}\), frequency \(\tau\), which together define the initial cotangent vector \((\zeta_0, \zeta_0) = -\tau c(z_0, x_0)^{-1}x\), and time \(t\),

\[
\eta(t, z_0, x_0, x, \tau) = \eta_x(t, z_0, x_0, x, \tau), \quad \eta_x(t, z_0, x_0, x, \tau), \quad \eta_x(t, z_0, x_0, x, \tau), \quad \eta_x(t, z_0, x_0, x, \tau).
\]

(2)

Here the evolution parameter is the time \(t\). Note that \(\tau\) is invariant along the Hamilton flow (see e.g. the introduction and chapter 5 of [9]).

Hörmander’s wave front set will be denoted by WF\((u)\), for a distribution \(u\) in \(\mathcal{D}'(\mathbb{R}^d)\) for some \(k [12, \text{chapter 8}]\). For the further analysis, we also need the following notion of microlocal equivalence. Let \(u, v \in \mathcal{D}'(\mathbb{R}^d)\). Then \(u \equiv v\), microlocally on \(\Gamma \subset T^*\mathbb{R}^d \setminus 0\), if \(WF(u - v) \cap \Gamma = 0\).

The data are assumed to be modeled by \(\delta G(z_r, r, t, z_s, s)\), for \(z_r = z_s = 0\). Here \(s \in \mathbb{R}^{n-1}\), \(r \in \mathbb{R}^{n-1}\) denote the horizontal source and receiver position, respectively. We have (cf. Paper I and [16])

\[
\delta G(0, r, t, 0, s) = \int dz \int dx \int dG(0, r, t, \ell', z, x) \frac{2\delta c(z, x)}{c_0(z, x)^3} \partial_t^2 G(z, x, \ell', 0, s).
\]

(3)

We assume that the data are available for \((s, r, t)\) in a bounded open subset \(Y\), called the acquisition manifold, of \(\mathbb{R}^{2n-2} \times \mathbb{R}_+\). Except for Remark 2.5 we assume that no further restrictions apply to \(Y\); the acquisition geometry is maximal. The modeling or scattering operator \(F\) is, for given \(c_0\), defined as the map from \(\delta c\) to \(\delta G\) restricted to \(Y\). Since \(Y\) is bounded and the waves propagate with finite speed we may assume that \(\delta c\) is supported in a bounded open subset \(X\) of \(\mathbb{R}^{n-1} \times \mathbb{R}_+\). We furthermore assume that \(\overline{X} \cap \{z = 0\} = \emptyset\).

Under certain conditions, the operator \(F\) is a Fourier integral operator [21] with canonical relation

\[
\{(\eta_x(t_v, z, x, \beta, \tau), \eta_x(t_v, z, x, \beta, \tau), t_v + t_v, \eta_x(t_v, z, x, \beta, \tau), \eta_x(t_v, z, x, \beta, \tau), \tau; z, x, \zeta, \zeta) | t_v, t_v > 0, \eta_x(t_v, z, x, \beta, \tau) = \eta_x(t_v, z, x, \beta, \tau) = 0, (\zeta, \zeta) = -\tau c_0(z, x)^{-1}(\alpha + \beta), (z, x, \alpha, \beta, \tau) \in \text{subset of } X \times (S^{n-1})^2 \times \mathbb{R} \setminus \{0\} \subset T^*\mathbb{R}^{2n-1} \times T^*\mathbb{R}^n, \}
\]

(4)

see also the discussion in the introduction of Paper I; for a general reference of Fourier integral operators, see [9]. We note that \(WF(\delta G)\) is contained in the composition of this canonical relation with \(WF(\delta c)\).

1.2. The downward continuation method

The main topic of this paper is the so-called wave-equation or downward continuation approach [5,4,19] to seismic inverse scattering. We summarize and discuss the results of Paper I. Key ingredients in this approach are an operator \(H(z, z_0)\) and its adjoint \(H(z, z_0)^*\), where for technical reasons \(z_0 > z\). Let the data be denoted by \(d = d(s, r, t)\). The result of acting with \(H(0, z)^*\) on \(d\) has the geophysical interpretation of the data that would have been measured if the plane containing sources and receivers would be at depth \(z\) (this interpretation is not exact, it ignores certain pseudodifferential factors affecting the amplitude in the data, and can only be made after restriction to positive times of \(H(0, z)^*d\). Thus \(H(z, z_0)\) and its adjoint are operators acting on distributions of \((s, r, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}\). In the following paragraphs we will discuss this in more detail. We start with so called one-way wave equations.

1.2.1. One-way wave propagation

Let \(C_{-\beta}\) be the cone \(C_{-\beta} = \{(v_x, v_x) | (v_x, v_x) \in \mathbb{R} \times \mathbb{R}^{n-1} \| v_x \| < \tan(\theta) | v_x | \text{ and } v_x < 0\}\). One-way wave equations describe only waves propagating with ray velocities, \((v_x, v_x) = \frac{d h}{dr}, \frac{d h}{dr}\), inside such a cone, where \(\theta < \exists\). Numerical one-way wave equations have been extensively studied, see e.g. [4,22], or [8] and references therein. For the analysis of such equations, see [25]. In this section, we summarize some of the aspects needed here.

The essential property of one-way wave equations is that they give solutions to the wave equation microlocally for singularities propagating along bicharacteristics that stay in \(C_{-\beta}\), for some given \(\theta\). To make this precise, we consider a bicharacteristic going through some point \((z_0, x_0, t_0, \zeta_0, \zeta_0, \tau)\), and assume that

(1) the associated velocity vector \(\frac{d z}{d r} \frac{d x}{d r}\) stays in \(C_{-\beta}\) along some open interval including \((z_0, x_0, t_0, \zeta_0, \zeta_0, \tau)\);
(2) \(u\) satisfies Eq. (1) with \(f\) satisfying \(f \equiv 0\) microlocally on this interval of the bicharacteristic;
(3) the function $f$ is in $C^\infty$ on a neighborhood of $(z_0, x_0, t_0)$ and $(z_0, x_0, t_0, -\zeta_0, \xi_0, \tau_0) \notin \text{WF}(u)$ (this would generate a singularity with vertical propagation velocity $v_z > 0$, i.e. in the opposite direction as those in $C_{-\theta_1}$).

Then $Q^* u_- \equiv u$ on this bicharacteristic, where $Q_- = Q_- (z, x, D_z, D_t)$ is an elliptic pseudodifferential operator with symbol $\text{sgn} (\tau) |\tau|^{-1/2} (c_0^2 - \tau^{-2}) |\xi_c|^2 |\xi_s|^2$, and $u_-$ is the solution to an initial value problem in $z$,

$$(\partial_z - iB_-) u_-, \quad z < z_0, \quad Q_- |_{z=z_0} u_-, \quad u|_{z=z_0}, \quad (5)$$

with $B_-$ a pseudodifferential operator with principal symbol given by

$$-b(z, x, \xi, \tau) = \tau \sqrt{c_0(z, x)} - \tau^{-2} |\xi_c|^2.$$

Here we use the notations $D_z = -i\partial_z$. Such $B_-$ and $Q_-$ exist for any fixed $\theta_1 < \frac{\pi}{2}$.

We assume that singularities propagating with velocity outside the cone $C_{-\theta_1}$, $\theta_1 < \theta_2 < \frac{\pi}{2}$, are strongly damped. This damping can be obtained by introducing a new term $Cu_-$ in the differential equation, where $C(z, x, D_z, D_t)$ is a suitably chosen pseudodifferential operator, see [25]; such an approach was followed in a finite-difference scheme for the paraxial approximation to the one-way wave equation in [29, IV.D]. We thus replace (5) by the one-way wave equation with damping,

$$(\partial_z - iB_- - C) u_-, \quad z < z_0. \quad (6)$$

Denote by $G_-(z, z_0)$ the evolution operator associated with this equation, mapping initial values at $z_0$ to the solution at $z$. Then $\ker Q_- (z, x, D_z, D_t) G_-(z, z_0) Q_- (z_0, x_0, D_{z_0}, D_{t_0})$ models microlocally the upgoing constituents of the solution operator to (1), in the sense that its distribution kernel is microlocally equal to $G(z, x, t, z_0, x_0)$.

Here $\ker$ denotes the Hilbert transform in time.

$G_-(z, z_0)$ propagates singularities along the bicharacteristics of $\partial_z - iB_-$; they are parameterized by $z$, initiated at $(z_0, x_0)$ and $\xi_0$, and denoted as $\gamma (z, x, t, z_0, x_0, \xi_0, \tau)$. In components, we write them as (note that they are time translation invariant)

$$\gamma (z, x, t, z_0, x_0, \xi_0, \tau) = (z, \gamma_x (z, x, t, z_0, x_0, \xi_0, \tau), \gamma_t (z, x, t, z_0, x_0, \xi_0, \tau) + t_0, -b(z, \gamma_x, \gamma_t, \tau), \gamma_z (z, x, t, z_0, x_0, \xi_0, \tau), \gamma_{\xi} (z, x, t, z_0, x_0, \xi_0, \tau), \gamma_{\tau} (z, x, t, z_0, x_0, \xi_0, \tau), \gamma_{s} (z, x, t, z_0, x_0, \xi_0, \tau), \gamma_{c} (z, x, t, z_0, x_0, \xi_0, \tau)). \quad (7)$$

1.2.2. Upward/downward continuation

In Paper I, we defined an operator $H(z, z_0)$, called the upward continuation operator, by a composition of two one-way wave Green’s functions. It acts on distributions of $(s, r, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$, and is given by

$$H(z, z_0) = ( \text{Id}_s \otimes G_{-\tau} (z, z_0) \otimes \text{Id}_s). \quad (8)$$

Here, $G_{-\tau} (z, z_0)$ acts in the $(s, t)$ variables, and $\text{Id}_s$ is the identity operator on functions of $s$, and a similar notation is assumed for $G_{-\tau} (z, z_0)$ and $\text{Id}_r$. We observed in Paper I that $G_{-\tau}$, $B_{-\tau}$, $C$, commute with $G_{-\tau}$, $B_{-\tau}$, $C$. From this it follows that $H(z, z_0)$ is the solution operator of the initial value problem for the so-called double-square-root (DSR) equation

$$\left( \frac{\partial}{\partial z} - iB_- (z, s, D_z, D_t) - iB_- (z, r, D_z, D_t) - C(z, s, D_z, D_t) - C(z, r, D_z, D_t) \right) u = 0, \quad 0 \leq z < z_0, \quad (9)$$

i.e., the solution to (9) with initial value $u (z_0, s_0, r_0, t_0) = \nu (s_0, r_0, t_0)$ is given by $H(z, z_0) \nu$. There are bicharacteristics associated with (9) and they are given by

$$\Gamma (z, z_0, s_0, r_0, t_0, s_0, \rho_0, \tau) = (\gamma_x (z, z_0, s_0, \sigma_0, \tau), \gamma_t (z, z_0, s_0, \rho_0, \tau), t_0 + \gamma_z (z, z_0, s_0, \sigma_0, \tau), \gamma_{\xi} (z, z_0, s_0, \sigma_0, \tau), \gamma_{\tau} (z, z_0, s_0, \sigma_0, \tau), \gamma_{s} (z, z_0, s_0, \sigma_0, \tau), \gamma_{c} (z, z_0, s_0, \sigma_0, \tau), \gamma_{t} (z, z_0, s_0, \sigma_0, \tau), \gamma_{s} (z, z_0, s_0, \sigma_0, \tau)). \quad (10)$$

The adjoint $H(0, z)^*$ is used to compute the downward continued data (for example, by method of generalized screen expansion [8]).

To justify the substitution for the time convolution of the two Green’s functions in (3) by $H$, we will need

**Assumption 1.** (DSR assumption) If $(z, x) \in X$ and $\alpha, \beta \in S^{n-1}$, $t_k, t_i > 0$ depending on $(z, x, \alpha, \beta)$ are such that $\eta_z (t_k, z, x, \beta, \tau) = \eta_z (t_i, z, x, \alpha, \tau) = 0$ and $(\eta_x (t_k, z, x, \beta, \tau), \eta_x (t_i, z, x, \alpha, \tau), t_k + t_i) \in Y$ (cf. (4)), then
\[ c(z,x)^{-1} \frac{\partial \eta}{\partial t} (t,z,x,\beta,\tau) < - \cos(\theta_1), t \in [0, t_s] \tag{11} \]
\[ c(z,x)^{-1} \frac{\partial \eta}{\partial t} (t,z,x,\nu,\tau) < - \cos(\theta_1), t \in [0, t_s]. \tag{12} \]

### 1.2.3. Factorization of the scattering operator

We will denote by \( F_D \) the scattering operator but with the one-way Green’s functions substituted for the Green’s functions (cf. (3)). In Section 5 of Paper I, we compared this scattering operator, in the downward continuation approach, with the original operator \( F \). To write \( F_D \) in convenient form, we first define two mappings,

\[ E_1 : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{2n-1}) : c(z,x) \mapsto \delta(x - \bar{x})c(\frac{z - x}{2}), \tag{13} \]
\[ E_2 : \mathcal{D}'(\mathbb{R}^{2n-1}) \to \mathcal{D}'(\mathbb{R}^{2n}) : h(z,\bar{x},x) \mapsto \delta(t)h(z,\bar{x},x), \tag{14} \]

and a mapping \( L : \mathcal{E}'(\mathbb{R}_{(x,x',t),s}) \to \mathcal{D}'(\mathbb{R}^{2n-1}) \) by

\[ Lg = Q'_{-s}(0)Q'_{-r}(0) \int_0^Z H(0,z)Q'_{-s}(z)Q'_{-r}(z)g(z,\cdot',\cdot')(s,r,t) \, dz. \tag{15} \]

It was shown that

\[ F_D = \frac{1}{2}c_0^{-3}\delta c \to D_1^2LE_2E_1(\frac{1}{c_0^{-3}} \delta c). \tag{16} \]

In the above, for given \( z, g = E_2E_1(\frac{1}{c_0^{-3}} \delta c) \) has the interpretation of kernel of a reflection operator. (Note that we have absorbed a multiplication by \( 2c_0^3 \) in \( F_D \) unlike the definition in Paper I.)

For the composition \( LE_2 \) we introduced the notation \( K \),

\[ K = LE_2. \tag{17} \]

The operators \( L \) and \( K \) are Fourier integral operators, see Paper I, Lemma 3.1 and Theorem 4.2. We found (Theorem 5.1) that there is a pseudodifferential operator \( \psi_D \) such that

\[ F_D(\frac{1}{c_0^{-3}} \delta c) \equiv \psi_DF \delta c \text{ or } F_D(\frac{1}{c_0^{-3}} \delta c) \equiv \psi_DF. \tag{18} \]

The operator \( \psi_D \) can be called a pseudodifferential cutoff, it is 1 on the set \( \Omega_{\theta_1} \), and in \( S^{-\infty} \) outside \( \Omega_{\theta_2} \), with \( 0 < \theta_1 < \theta_2 < \pi/2 \) and where \( \Omega_{\theta} \) is given by

\[ \Omega_{\theta} = \{ (s,r,t,\sigma,\rho,\tau)| |t < T_{\max}(0,s,r,\sigma,\rho,\tau,\theta) \}. \tag{19} \]

Here the maximal time \( T_{\max} = T_{\max}(0,s,r,\sigma,\rho,\tau,\theta) \) is determined by the canonical relation of \( K \) from the maximal depth, \( Z_{\max} = Z_{\max}(0,s,r,\sigma,\rho,\tau,\theta) \), that is the upper boundary of the maximal interval containing \( z = 0 \) for which the source and receiver rays satisfy the DSR assumption at \( \theta = \theta_1 \).

### 1.3. The reconstruction of \( \delta c \) given \( c_0 \)

There are a number of results concerning the approximate reconstruction of \( \delta c \) given \( c_0 \). Those differ in the assumptions on the dimension of the acquisition manifold and on the behavior of the rays (e.g. whether caustics are allowed) [1,11,17,16,23]. The approach common in these works is to first establish that the scattering operator is a Fourier integral operator and determine its canonical relation, and then to introduce a trial inverse scattering operator with the transpose canonical relation, but with a symbol that needs to be determined. This approach stems from the notion of imaging.

We introduce a function \( \psi_D \in C_0^\infty(Y) \) that goes smoothly to zero near the boundary of \( Y \). Applying the imaging after modeling yields the composition of operators \( N := F^*\psi_YF \), called the normal operator. The Bolker condition of Guillemin [10] is
Assumption 2. The projection of the canonical relation (4) on $T^*\mathbb{Y}\setminus 0$ is an embedding.

Since (4) is a canonical relation that projects submersively on the subsurface variables $(z, x, \zeta, \xi)$, the projection of (4) on $T^*\mathbb{Y}\setminus 0$ is immersive [14, 25.3.6]. Therefore only the injectivity in the assumption needs to be verified [16].

**Theorem 1.1.** With Assumption 2 the operator $N$ is a pseudodifferential operator.

Applying the adjoint $F^*\psi_\gamma$ to the data $d$ yields the normal equation,

$$N(z, x, D_z, D_x)\delta c = F^*\psi_\gamma d.$$  \hfill (20)

Thus, the problem becomes to invert this normal operator. Microlocally the inverse of $N$ exists, where its principal symbol is nonzero. The subset of $T^*\mathbb{R}_{(z,x)}^n \setminus 0$ where this is the case is determined as follows. The principal symbol of the normal operator is nonzero at $(z, x, \zeta, \xi)$ whenever there is a point $(s, r, t, \sigma, \rho, \tau; z, x, \zeta, \xi)$ in the canonical relation (4) with $(s, r, t, \sigma, \rho, \tau)$ in the support of $\psi_\gamma$. The mentioned subset is representative for the “illumination” of $\delta c$ by the available data.

In the reconstruction of $\delta c$, seismologists distinguish between imaging, which produces a function that has singularities at the same position as the ones in $\delta c$, and methods that also correctly compute the size of the singularities or the discontinuities. In the latter case they speak of inversion, or of true-amplitude imaging. Thus $F^*\psi_\gamma$ is an imaging operator, while $N^{-1}F^*\psi_\gamma$ is an inversion operator, where $N^{-1}$ is the regularized parametrix of $N$, that is an inverse microlocally for a subset of $T^*\mathbb{R}_{(z,x)}^n$ where the symbol of $N$ is nonzero. The inversion operator reconstructs microlocally $\delta c$.

The first result of this paper concerns the reconstruction of $\delta c$ in the downward continuation approach. In Section 2 we derive the normal equation in this approach from the Born DSR modeling operator, $F_D$, and derive a reconstruction equation in Theorem 2.2. This theorem is the downward continuation counterpart of Theorem 1.1. The adjoint $F_D^*$ is factorized into a product of operators to emphasize the different steps that compose an algorithm for reconstruction. It is anticipated that, because of the cutoff $\psi_\Omega$ in (18), fewer singularities in $\delta c$ will be reconstructed than if the original operator $F^*$ were used for the imaging. For example, vertical reflectors that could be illuminated by turning rays are not reconstructed. In many practical cases this disadvantage is however not so important.

### 1.4. A processing method for identifying acceptable $c_0$

The results of Section 3 concern the determination of $c_0$ through migration velocity analysis. Here it is exploited that the reconstruction of $\delta c$ is an overdetermined problem. Indeed $\delta c$ is a function of $n$ variables, and $d$ of $2n-1$ variables. Beylkin [1] gives conditions so that $\delta c$ can be reconstructed by an operator $A_{ss}$, say, from a subset of dimension $n$ of the data, given $c_0$. An $n$ dimensional subset of data are obtained, for instance, by taking the source coordinate constant. In conventional migration velocity analysis the data are viewed as an $(n-1)$-parameter family of such subsets, which results in an $(n-1)$-dimensional family of reconstructions. If we denote the $(n-1)$ parameters by $p$, and the family of reconstructions of $\delta c$ by $A_{ss}[c_0]d(z, x, p)$, then this gives the following criterion for the determination of $c_0$:

$$\langle A_{ss}[c_0]d \rangle(z, x, p) \text{ is independent of } p,$$  \hfill (21)

at least microlocally. In migration velocity analysis some $c_0$ is constructed based on this criterion. Note that $c_0$ need not be uniquely determined by the singular part of the data, that is why we say “acceptable” $c_0$.

In the presence of caustics the conditions of [1] are generally violated, and the application of $A_{ss}$ (which can still be defined in this case) results in images with so called kinematic artifacts, that correspond to *nonlocal singular contributions* to $A_{ss}F$ [17,18]. As a result, (21) is no longer valid. To remedy this, the parameter $p$ can be chosen to no longer parameterize a subset of data, but instead to parameterize the angle between in- and out-going rays at the image point. In the Kirchhoff approach to seismic inverse scattering it has been proposed to use a generalized Radon transform to generate a set of images parameterized by angle (such an operator will be referred to as angle transform) [3,33]. However, in the presence of caustics, artifacts were identified in numerical examples [3,28], also in this case. Such artifacts are attributed to contributions from
nonlocal operators to the composition of the generalized Radon transform with the modeling operator \( F \). By microlocal analysis of the Kirchhoff approach the presence of artifacts was shown in [24].

The second and main result of this paper is to show that there is a downward continuation based angle transform, \( A_{\text{WE}} \), that is artifact free under much weaker conditions than those of Beylkin [1] (Theorem 3.1). In particular, the presence of caustics is allowed. The map \( A_{\text{WE}} \) is defined in Section 3, and resembles an operator introduced in [6] to study angle dependent reflection coefficients. In (16) we observe that

\[
E_d \left( F_{\text{WE}} \delta c \right) = \left( 2\pi \right)^{-\frac{n}{a}} \int \left( \frac{c_0^{-3}}{2} \delta c \right) \left( \frac{x + t}{2} \right) \exp \left( i \left( x - x \right), p \right) \exp \left( i \tau \right) |\tau|^{n-1} d\tau
d\tau
\]

where \( A \) is a pseudodifferential operator with symbol \(|\tau|^{n-1}\), and \( E_3 : \mathcal{D}'(\mathbb{R}^{2n-1}) \rightarrow \mathcal{D}'(\mathbb{R}^{2n})\), \( E_4 : \mathcal{D}'(\mathbb{R}^{n}) \rightarrow \mathcal{D}'(\mathbb{R}^{2n-1})\) are given by

\[
E_3 f(x, x, t) = \left( 2\pi \right)^{-(n-1)} \int \delta(t - (x - x), p) f \left( z, \frac{x + t}{2}, p \right) dp,
\]

\[
E_4 \left( \frac{c_0^{-3} \delta c}{} \right) (z, x, p) = \left( \frac{1}{2} c_0^{-3} \delta c \right) (z, x),
\]

respectively, so that

\[
F_{\text{D}} = D_{\text{L}}^2 L E_3 E_4.
\]

By the map \( E_4 \) the perturbation \( \frac{1}{2} c_0^{-3} \delta c \) is viewed as being \( p \) dependent as if it were a reflection coefficient. The angle transform \( A_{\text{WE}} \) is derived, in the imaging process, from the adjoint \( E_3^* AL^* \). In the above, \( p \) is, given the singular direction from the wavefront set of \( \delta c \), obtained through trigonometric formulae related to scattering angles [7].

We show that \( A_{\text{WE}} F \) is a \( p \)-family of pseudodifferential operators, which implies that nonlocal singular contributions are absent and data are mapped to a set of images (Theorem 3.1). In addition we show that \( A_{\text{WE}} \) is an invertible Fourier integral operator. We introduce an appropriate modification, denoted by \( A_{\text{WE}} = A_{\text{WE}}[c_0] \) such that \( A_{\text{WE}}[c_0] F[c_0] \) is a \( p \)-family of pseudodifferential operators with symbol 1, microlocally, hence mapping data to a set of reconstructions (Proposition 3.2). In [7] it was confirmed numerically that this angle transform does not generate artifacts in the presence of caustics.

The third result concerns the construction of operators we will refer to as annihilators of the data. The notion of acceptable background model \( c_0 \) can now be made precise by requiring that the data are in the range of the modeling operator \( F_{\text{D}}[c_0] \). The criterion that the data \( d \in \text{range}(F_{\text{D}}[c_0]) \) becomes equivalent to \( (A_{\text{WE}}[c_0]d)(z, x, p) \) is independent of \( p \),

\[
(\bar{A}_{\text{WE}}[c_0])d(z, x, p) \text{ is independent of } p,
\]

microlocally, where \( p \) can be identified with the integration variables in \( E_3 \) (cf. (23)). Thus, the criterion for migration velocity analysis (21) is extended to allow for background media with caustics.

Eq. (26) reveals the redundancy in the data. The existence of pseudodifferential operators that annihilate the singular part of the data are associated with this redundancy [26]. The annihilators are related to the differential semblance measure in the framework of Beylkin’s conditions, for estimating the background medium [30]. Annihilators of the data in the downward continuation approach can be derived from \( A_{\text{WE}} \). In addition, we construct in Section 4 a different annihilator, \( W \), such that \( ||Wd||_{L^2} \) measures the focusing at \( r = s \) of the downward continued data restricted to \( t = 0 \).

2. Imaging and reconstruction in the downward continuation approach

Conceptually, the first step in the reconstruction of the perturbation \( \delta c \) given the background, \( c_0 \), is applying the adjoint of the linear modeling map to the data. This is the process of imaging. We present a normal equation similar to (20) based on the double-square-root (DSR) modeling operator (16). The operator \( F^* \) in the right hand side of (20) is replaced by \( F_{\text{D}}^* \), so that the right hand side can be computed using the downward
continuation approach summarized in the introduction. As in Theorem 1.1 and the discussion below it, this leads to reconstruction modulo a pseudodifferential operator for which an explicit expression is given.

First, we discuss the geometric or kinematic considerations pertaining to the reconstruction and the prevention of imaging artifacts. The operator $F$ is a Fourier integral operator whose canonical relation is a subset of that of $D$ containing the elements with $(z, x, \alpha, \beta)$ such that Assumption 1 applies, with angle given by $\theta_2$ below (18). It follows that in this case Assumption 2 can be replaced by the following weaker assumption

**Assumption 3.** If several elements of (4) project to the same point in $T^* Y$, then none of the $(z, x, \alpha, \beta)$ that describe these elements are such that Assumption 1 is satisfied.

In effect this assumption implies that the cutoff $\psi_Y$ has symbol in $S^{-\infty}$ for points in $T^* Y$ where the projection from (4) is not injective. A cartoon of two ray pairs such that Assumption 3 is violated is given in Fig. 1. As discussed in Paper I, Section 6, the DSR Assumption 1 is stronger than Assumption 2, so is also sufficient for Theorem 1.1 to hold.

Second, we discuss a factorization of $F_D^*$, and the different adjoints of operators that make up this factorization. In this way we obtain insight in the structure of the imaging operator and in the different steps in algorithms to implement it.

Consider the factorization of $F_D$ given in Eq. (16). The adjoint operator $H(0, z)^*$ propagates the data downward and backward in time. The adjoint of operator $L$ is given by

$$
(L^* d)(z, s, r, t) = Q_{-s}^*(z)Q_{-r}^*(z)H(0, z)^* Q_{-s}(0)Q_{-r}(0)d,
$$

which yields, in general, a nonvanishing outcome for $t < 0$ also. The adjoint of extension operator $E_2$ is given by the restriction $R_2$ defined by

$$
g(z, s, r, t) \rightarrow (R_2 g)(z, s, r) = g(z, s, r, 0),
$$

while the adjoint of extension operator $E_1$ is given by the restriction $R_1$ defined by

$$
h(z, s, r) \rightarrow (R_1 h)(z, x) = h(z, x, x).
$$

In any algorithm derived from the inverse scattering approach developed in this paper, the operator $L^*$ is the key component; the computation of $L^* d$ would be implemented as a ‘marching’ in depth $z$. The restrictions $R_1, R_2$ represent the imaging conditions and are applied at each depth after the data are downward continued.

The adjoint of the depth-to-time conversion operator, $K = L E_2$ equals

$$
K^* = R_2 L^*.
$$

The canonical relation of $K$ maps points $(z, s, r, \zeta, \sigma, \rho)$ in a subset of $T^* \mathbb{R}^{2n-1} \setminus 0$ diffeomorphically to points $(s_0, r_0, t_0, \sigma_0, \rho_0, \tau)$ in a subset of the cotangent acquisition space $T^* \mathbb{R}^{2n-1} \setminus 0$ (Theorem 4.2 of Paper I). We denote this map by $\Sigma$. The adjoint $K^*$ maps singularities according to the inverse $\Sigma^{-1}$. Furthermore, for given $(z, s, r, \sigma, \rho)$, we have a mapping $\tau \mapsto \zeta = \Theta(z, s, r, \sigma, \rho)$ with inverse $\tau = \Theta^{-1}(z, s, r, \zeta, \sigma, \rho)$ (Lemma 4.1 of Paper I).

Third, we analyze the normal operator derived from $F_D$. In the downward continuation version of (20), the operator on the left hand side is given by $F_D^* \psi_Y F$. Here $\psi_Y = \psi_Y(s, r, t)$ is a smooth cutoff function on $Y$ that is

![Fig. 1. Cartoon of two ray pairs such that Assumption 3 is violated. The two trajectories between the reflection points are assumed to have equal traveltime.](image-url)
zero near the boundary of $Y$, that we introduced to account for the limited acquisition aperture. Suppose now that $K'$ is defined as $K$, but with angles for the cutoff (contained in $G_{-,\sigma}, G_{-,r}$) given by $\theta_1', \theta_2'$ instead of $\theta_1, \theta_2$, such that $0 < \theta_1 < \theta_2 < \theta_1' < \theta_2' < \pi/2$. Then $K$ is supported in the region where the cutoff that forms part of $K'$ is equal to 1. Define an operator $F_D'$ by $F_D' = D^2 K' E_1$ (cf. (16)-(18)). Then $F_D$ can be written as $F_D = \psi_D F'_D$, modulo a regularizing operator. Hence

$$F_D^* \psi_D F = F_D^* \psi_D F D^2 \left(\frac{1}{2} \chi_0^{-1}\right),$$

modulo a regularizing term. Thus, we can compute the symbol of $F_D^* \psi_D F$ using the downward continuation expression for $F$. Substituting (16), we have that

$$F_D^* \psi_D F_D = R_1 K^* D^2 \psi_D D^2 K E_1.$$  

The first step in our computation of the symbol of this operator will be to evaluate $K^* D^2 \psi_D D^2 K$. To simplify the evaluation, we introduce intermediate operators $L$ and $K$ that correspond with operators $L$ and $K$, respectively, with the $Q_-$ operators removed,

$$K = LE_2$$

with

$$Lg = \int_0^z H(0,z)g(z,\cdot,\cdot) \, dz.$$  

We note, however, that $Q_{-\sigma}$ and $Q_{-\sigma}$ do not commute with $E_2$.

**Lemma 2.1.** The composition $K^* D^2 \psi_D D^2 K$ is a pseudodifferential operator of order 0 with principal symbol $\Sigma'(\psi|D) (z, s, r, \zeta, \sigma, \rho)$ $\Xi(z, s, r, \zeta, \sigma, \rho)$, where $\Xi$ is given by

$$\Xi(z, s, r, \zeta, \sigma, \rho) = (\Sigma^* a_0)(z, s, r, \zeta, \sigma, \rho) \times \alpha_c(s, r, \sigma, \rho, \Theta^{-1}(z, s, r, \zeta, \sigma, \rho)) \Xi(z, s, r, \zeta, \sigma, \rho)$$

where

$$\alpha_c(s, r, \sigma, \rho, \zeta) = \tau^2 |b(z, s, \sigma, \tau)|^{-1} |b(z, r, \rho, \zeta)|^{-1}$$

and

$$\Xi(z, s, r, \zeta, \sigma, \rho)^{-1} = \left[ \frac{\partial \Theta}{\partial r}(z, s, r, \sigma, \rho, \Theta^{-1}(z, s, r, \zeta, \sigma, \rho)) \right]$$

$$= \left[ c_0(z, s)^{-2} (c_0(z, s)^2 - \tau^{-2} |\sigma|^2)^{-1/2} + c_0(z, r)^2 - \tau^{-2} |\rho|^2)^{-1/2} \right]_{r=\Theta^{-1}(z, s, r, \zeta, \sigma, \rho)}.$$  

**Proof.** From the fact that the canonical relation of $K$ is invertible (the graph of a diffeomorphism denoted by $\Sigma$ before) between subsets of $T^* \mathbb{R}^{2n-1} \setminus 0$, it follows that $K^* \psi_D K$ is a pseudodifferential operator. We calculate the principal symbol of $K^* D^2 \psi_D D^2 K$.

First, we evaluate the symbol of $\bar{K} \bar{K}$ microlocally, ignoring the cutoffs; its principal part we denote by $\Xi(z, s, r, \zeta, \sigma, \rho)$. We recall that the kernel of operator $H(0,z)$ has microlocally an oscillatory integral representation with amplitude $A = A(z,y_0, \eta_0, s, r, t)$ and a phase function associated with generating function, $S = S(z,y_0, \eta_0, s, r, t)$, where $y_0 = (s_0, r_0, t_0)$ and $\eta_0$ is the corresponding cotangent vector, and $(I,J)$ is a partition of $\{1, \ldots, 2n-1\}$ (Lemma 3.1 of Paper I). The kernel of operator $K$ has an oscillatory integral representation similar to the one of $H(0,z)$,

$$K(y_0,z,s,r) = (2\pi)^{-\left(2n-1+|I|\right)/2} \int A(z,y_0, \eta_0, s, r, t) \exp\left(iS(z,y_0, \eta_0, s, r, 0) + (\eta_0, y_0)\right) \, d\eta_0$$

which follows upon considering the action of $\bar{L}$ on $\delta(t)$, distribution $(z, s, r)$ and carrying out the $t$ integration. (The factors of $2\pi$ follow the convention used, for example, in [9].)
We evaluate, microlocally, $\tilde{K}^*\tilde{K}$. Using (38) and integrating out one set of $\eta_{0j}$ variables, the kernel of $\tilde{K}^*\tilde{K}$ can be written as

$$
(2\pi)^{-(2n-1)} \int A(z',y_0,\eta_{0j},s',r',0) A(z,y_0,\eta_{0j},s,r,0) \exp(i[-S(z',y_0,\eta_{0j},s',r',0) + S(z,y_0,\eta_{0j},s,r,0)]) \, d\eta_{0j} \, dy_0
$$

(39)

Next we identify the gradient

$$
-\frac{\partial S}{\partial(z,y_0,\eta_{0j},s,r,0)} (z,y_0,\eta_{0j},s,r,0) = (\zeta(z,y_0,\eta_{0j},s,r,0), \sigma(z,y_0,\eta_{0j},s,r,0), \rho(z,y_0,\eta_{0j},s,r,0)).
$$

(40)

Applying a change of variables of integration, $(y_0,\eta_{0j}) \mapsto (\zeta, \sigma, \rho)$, the phase in the oscillatory integral representation of the kernel of $\tilde{K}^*\tilde{K}$ takes the form

$$
\langle (\zeta, \sigma, \rho), (z' - z, s' - s, r' - r) \rangle;
$$

(41)

a Jacobian, $[(\frac{\partial (\zeta, \sigma, \rho)}{\partial(y_0,\eta_{0j})})]^{-1}$, appears in the amplitude, so that the amplitude of $\tilde{K}^*\tilde{K}$ has principal part (using the expression for $A$ from Lemma 3.1 of Paper I)

$$
\left| \frac{\partial (\zeta, \sigma, \rho)}{\partial(y_0,\eta_{0j})} \right|^{-1} \frac{\partial (\sigma, \rho, \tau)}{\partial(y_0,\eta_{0j})} \bigg|_{\tau = 0}.
$$

(42)

For fixed $(z,s,r,\sigma,\rho)$ the map $\tau \mapsto \zeta = \Theta(z,s,r,\sigma,\rho,\tau)$ is invertible on a set given by $|\tau|$ sufficiently large: $\tau = \Theta^{-1}(z,s,r,\zeta,\sigma,\rho)$ (Lemma 4.1 of Paper I). Carrying out the multiplication of determinants, it follows that the principal part of the symbol of $\tilde{K}^*\tilde{K}$ is microlocally given by

$$
\Xi(z,s,r,\zeta,\sigma,\rho) = \left| \frac{\partial \Theta}{\partial \tau}(z,s,r,\sigma,\rho,\Theta^{-1}(z,s,r,\zeta,\sigma,\rho)) \right|^{-1}.
$$

(43)

To obtain $K^*D^1_t\psi_yD^2_rK$ involves the composition of $\tilde{K}^*\tilde{K}$ with pseudodifferential operators. First, the composition

$$
\tilde{K}^*Q^{-}_t(0)Q^{-}_s(0)D^2_t\psi_yD^2_rQ^{-}_t(0)Q^{-}_s(0)\tilde{K}
$$

(44)

is carried out with the aid of Egorov’s theorem. This leads to a factor $\Sigma^*(r^2\psi_ya_0)$ in the principal symbol (for the definition of $\Sigma$, see the text below (30)). Secondly, having obtained a pseudodifferential operator (in (44)), the inclusion of the factors $Q^{-}_t(z), Q^{-}_s(z)$ and $Q^{-}_t(\zeta), Q^{-}_s(\zeta)$ in between $L$ and $E_2$ is carried out with the standard calculus of pseudodifferential operators. This leads to a factor $(\Theta^{-1}(z,s,r,\zeta,\sigma,\rho))^{-2} a_2(s,r,\sigma,\rho,\Theta^{-1}(z,s,r,\zeta,\sigma,\rho))$ in the principal symbol. It follows that the principal part of the symbol of $K^*D^1_t\psi_yD^2_rK$ is microlocally given by $\Xi \Sigma^* \psi_y$. With the pseudodifferential cutoff $\psi_D$ taken into account we obtain the result of the lemma.

We now determine the symbol of the normal operator in (32) and develop the downward continuation analogue of (20). To this end, we define an operator $\Psi$ using the operator $K'$ introduced above (31), by $\Psi = K' = D^2_r\psi_D^*D^2_t\psi_D^*K'$. It follows from the lemma that $\Psi$ is pseudodifferential in $S^0_p(\mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1})$ (for some $\rho$, $\frac{1}{2} < \rho < 1$, see Section 2 of Paper I) with principal symbol, that we denote by $\Psi_0$, given by

$$
\Psi_0(z,s,r,\zeta,\sigma,\rho) = \Xi(z,s,r,\zeta,\sigma,\rho)(\Sigma^*(\psi_D^*\psi_y))(z,s,r,\zeta,\sigma,\rho),
$$

(45)

in which the second factor reveals the illumination. Starting from the DSR Born modeling (16), we obtain the following reconstruction result.

**Theorem 2.2.** Suppose Assumption 3 is satisfied. Then there is a pseudodifferential operator $\Phi = \Phi(z,x,D_z,D_x)$ of order $n - 1$, $\Phi \in S^0_p(\frac{1}{2} < \rho < 1)$, with principal symbol

$$
\Phi_{n-1}(z,x,\zeta,\xi) = \int_{\mathbb{R}^{2n-1}} \Psi_0(z,x,x,\zeta,\xi,\frac{1}{2} \xi - \frac{1}{2} \xi + 0) \, d\theta,
$$

(46)
such that

$$\Phi(z, x, D_z, D_x)\left(\frac{\delta c}{2\pi}e^{-i\delta c}\right) = R_1K^*D^2\psi_1 d,$$

where $d = F\delta c$ is the Born modeled data (16).

**Proof.** Because of Theorem 1.1 and the modeling formula (16), the right-hand side of (47) is equal to a pseudodiffferential operator acting on $\frac{\delta c}{2\pi}$,$$
R_1K^*D^2\psi_1 d = R_1\Psi E_1\left(\frac{\delta c}{2\pi}\right).
$$

It remains therefore to compute the symbol of the composition of operators $R_1\Psi E_1$, cf. (31). The kernel of this composition has an oscillatory integral representation

$$(2\pi)^{-2(n-1)}\int_{\mathbb{R}^{2n-1}} \Psi(z, x, x, \zeta, \sigma, \rho) e^{i((z, x)^{(x', x'), (\zeta, 0))} \rho d\sigma d\zeta}$$

$$= (2\pi)^{-2(n-1)}\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \Psi(z, x, x, \zeta, \frac{1}{2}z - \frac{1}{2}z + 0) \rho d\sigma d\zeta,$$

upon changing variables of integration, $\sigma = \frac{1}{2}z - \frac{1}{2}z$, $\rho = \frac{1}{2}z + 0$. The support of the integrand in the inner integral is bounded by an inequality of the form $|z| \leq C(|\zeta| + 1)$, due to the cutoff $\psi_D$. Using this we next show that the inner integral is a symbol in $S^{-n-1}_{\rho} (\mathbb{R}^n \times \mathbb{R}^n)$. We have the inequality

$$\frac{\partial^a}{\partial (z, x)^a} \frac{\partial^b}{\partial (\zeta, \sigma)^b} \int_{\mathbb{R}^{2n-1}} \Psi(z, x, x, \zeta, \frac{1}{2}z - \frac{1}{2}z + 0) \rho d\sigma d\zeta$$

$$\leq C(|\zeta| + 1)^{-1} \max_{|\rho| < C(|\zeta| + 1)} \frac{\partial^a}{\partial (z, x)^a} \frac{\partial^b}{\partial (\zeta, \sigma)^b} \Psi(z, x, x, \zeta, \frac{1}{2}z - \frac{1}{2}z + 0),$$

(49)

so that the symbol estimates follow from the symbol estimates for $\Psi$.

From this symbol property it follows that $R_1\Psi E_1$ is a pseudodifferential operator of order $n-1$, with principal symbol (46).

Applying the above results to the expression (16) for the Born modeled data leads to the statement of the theorem. $\square$

The symbol of $\frac{\delta c}{2\pi}F_D^*\psi_1 F = F^*\psi_D^*\psi_1 F$ also follows from the computation of $F^*F$ of Ten Kroode et al. [16, theorem 4.1], upon inserting $\psi_D(s, r, t, \sigma, \rho, \zeta)\psi_1(s, r, t)$ in their formula (64). Our purpose was to show that this computation is independent of the results of [16]. Also, our method is easily modified for the result (55) below.

The right-hand side of (47) produces an image of (the singularities of) $\delta c$; to obtain an estimate of $\delta c$ requires the construction and application of the parametrix of $\Phi$ microlocally.

**Remark 2.3.** Note that in (47) the operator $\Phi(z, x, D_z, D_x)$ depends on the Hamiltonian flow associated with the background medium in the depth interval $[0, z]$. To account for the principal part of this operator, cf. (45) and (46), one requires a ray computation to determine $\Sigma^*$ separate from the downward continuation of the data with $K^*$.

**Remark 2.4.** Theorem 2.2 follows a least-squares data fitting approach to linearized inverse scattering. However, departing from this approach, we can derive a reconstruction equation the pseudodifferential operator in which only accounts for illumination effects. By the calculus of Fourier integral operators, the operator $K$ can be written as $K = Q_{-r}(0)Q_{-r}(0)K V$ modulo a smoothing operator, where $V$ is an operator with principal symbol

$$V(z, s, r, \zeta, \sigma, \rho) = |\tau|^{-1} (c_0(z, s)^2 - \tau^{-2}||\sigma||^2)^{-1/4} (c_0(z, r)^2 - \tau^{-2}||\rho||^2)^{-1/4}$$

We replace the normal equation in the downward continuation approach by the following equation,

$$K^*KE_1\left(\frac{\delta c}{2\pi}\right) = K^*Q_{-r}(0)^{-1}Q_{-r}(0)^{-1}D^2 f d.$$

(50)
Applying $V^{-1}\bar{\Xi}^{-1}$, we obtain the equation
\[ V^{-1}\bar{\Xi}^{-1}\tilde{K}'\tilde{K}V E_1(\frac{1}{\sqrt{c_0}}\delta c) = V^{-1}\bar{\Xi}^{-1}\tilde{K}'Q'_{\gamma,r}(0)^{-1}Q^*_{\gamma,r}(0)^{-1}D_t^{-2}d, \]  
(51)

microlocally, which reduces by (38)--(43) in the proof of Lemma 2.1 to
\[ \Psi E_1(\frac{1}{\sqrt{c_0}}\delta c) = V^{-1}\bar{\Xi}^{-1}\tilde{K}'Q'_{\gamma,r}(0)^{-1}Q^*_{\gamma,r}(0)^{-1}D_t^{-2}\psi_t d, \]  
(52)
in which $\Psi$ is a pseudodifferential operator with principal symbol $\bar{\Psi}_0 = \Sigma^r(\bar{\psi}_D\psi_t)$ (compare (45)). Therefore
\[ R_1 \Psi E_1(\frac{1}{\sqrt{c_0}}\delta c) = R_1V^{-1}\bar{\Xi}^{-1}\tilde{K}'Q'_{\gamma,r}(0)^{-1}Q^*_{\gamma,r}(0)^{-1}D_t^{-2}\psi_t d. \]  
(53)

Substituting $\bar{\Psi}$ for $\Psi$ in (46) then leads to the reconstruction equation
\[ \Phi(z, x, D_z, D_x) \left(\frac{1}{2}c_0^{-3}\delta c\right) = R_1V^{-1}\bar{\Xi}^{-1}\tilde{K}'Q'_{\gamma,r}(0)^{-1}Q^*_{\gamma,r}(0)^{-1}D_t^{-2}\psi_t d \]  
(54)

with
\[ \Phi(z, x, \zeta, \bar{\zeta}) = \int_{\mathbb{R}^{1\_1}} \bar{\Psi}(z, x, x, \zeta, \zeta, \frac{1}{2} - \theta, \frac{1}{2} + \theta) \, d\theta. \]  
(55)

The principal symbol of $\Phi(z, x, D_z, D_x)$ can in general be zero for some parts of phase space, due to aperture effects (see e.g. chapter 4 of [2]) so that a regularized inverse is required for the final reconstruction of $\frac{1}{2}c_0^{-3}\delta c$ (microlocally where $\Phi(z, x, D_z, D_x)$ is invertible). The symbol $\Psi$ contains a factor $\bar{\Xi}$, which is simpler than the factor $\Xi$ contained in $\Psi$, because of the absence of the pullback $\Sigma^r a_0$, compare (35) and (37). So the expression for $\Phi$ is simpler than the expression for $\Psi$, and therefore we prefer this equation for the development of a practical algorithm.

Remark 2.5. Depending on the background medium, the reconstruction can also be done using data on a submanifold $Y'$ of $Y$. Let $R'$ be the restriction of a function on $Y$ to $Y'$, so that the forward map for this case is given by $R'F$. In suitable local coordinates $(y', y'')$ on $Y$ such that $y'' = 0$ defines $Y'$, the adjoint $E'$ of $R'$ is given by the map $(E'f)(y', y'') = f(y')\delta(y'')$. Conditions such that $F'E'\psi, R'F$ is pseudodifferential are given in [17]. Reconstruction modulo a pseudodifferential operator is done in this case by first applying the map $E'$ to the data, and then applying the previous procedure. Applying $E'$ to the data simply means adding zeroes where there is no data in $Y$.

3. The wave-equation angle transform, common-image-point gathers

A method to generate multiple images of the medium contrast from the data follows from beamforming, here denoted by $R_3$, applied to the downward continued data, $g$ say,
\[ R_3 : g(z, s, r, t) \rightarrow (R_3g)(z, x, p) = \int_{\mathbb{R}^{1\_1}} g(z, x - \frac{h}{2}, x + \frac{h}{2}, (p, h)) \, dh. \]

Indeed, with the aid of (22) and (23), identifying $R_3$ with the adjoint of $E_3$, the modeling operator can be written in the form $F_D = D_t^3 LAE_3 E_4$ (cf. (25)). Following (partly) the process of imaging, that is, taking adjoints, we define the wave-equation angle transform, $A_{WE}$, containing a cutoff function $\chi$, as
\[ A_{WE} = R_3\chi L^* D_t^2, \]  
(56)

where the composition of $\chi$ and $R_3$ is
\[ R_3\chi : g(z, s, r, t) \rightarrow (R_3\chi g)(z, x, p) = \int_{\mathbb{R}^{1\_1}} g(z, x - \frac{h}{2}, x + \frac{h}{2}, (p, h)) \chi(z, x, h) \, dh, \]  
(57)
where $\chi(z, x, h)$ is a compactly supported cutoff function the support of which contains $h = 0$. For each $x$, the function $(z, p) \rightarrow (A_{WE}\psi_t d)(z, x, p)$ is a so-called common-image-point gather.
The work in this section was motivated by the papers [6] and [20], where a map similar to $A_{WE}$ was introduced. In the paper [6] the purpose was imaging of reflectors with angle-dependent reflection coefficients. The paper [20] suggested that this map could be suitable to obtain common-image point gathers in the presence of multipathing, which is indeed what we show here.

First, we analyze the properties of $A_{WE}$ mostly from a geometrical perspective. Then we derive a $p$-dependent reconstruction equation that replaces (47) or (54).

**Theorem 3.1.** Suppose Assumption 1 holds. Let $C_0$ be an upper bound for $c_0$. Assume that

$$
\|p\| < p_{\max} < C_0^{-1}.
$$

(58)

Then $A_{WE}$ is a Fourier integral operator such that $A_{WE} F$ is a smooth $p$-family of pseudodifferential operators in $(z, x)$. Let $C_1$ be an upper bound for $\frac{C_1}{C_0}$, $C_2$ an upper bound for $c_0^{-1}$. If in addition the function $h \mapsto \chi(z, x, h)$ of (56), contained in $A_{WE}$, is supported in $B(0, R)$, where $R$ depends on $\theta_2, C_0, C_1, C_2$, then the canonical relation of $A_{WE}$ corresponds to an invertible map from a subset of $T^*\mathbb{R}^{2n-1}_{(z,x,p)}$ to a subset of $T^*\mathbb{R}^{2n-1}_{(z,x,p)}$ that has nonempty intersection with the set $\vartheta = 0$ (where $\vartheta$ denotes the $p$-covector).

**Proof.** By Assumption 1 we have that $F = F_D(\frac{\omega^3}{\sqrt{3}^3})$, modulo a regularizing term. So for the first statement it is sufficient to show that $A_{WE} F_D$ is a $p$-family of pseudodifferential operators (pseudodifferential operators depending on a parameter $p$).

The Schwartz kernel of the map $R_{\partial t}$ equals

$$
\delta\left(x - s + \frac{r}{2}\right) \delta(p, r - s) \delta(z - z') \chi(z, x, r - s) = (2\pi)^{-n-1} \int \chi(z, x, r - s) e^{i(\xi z + x - \frac{p}{2}r) + r(p - r) + (z - z')} \, d\xi \, dt \, dz.
$$

(59)

It defines a Fourier integral operator with canonical relation

$$
\left\{ \left( \frac{s + r}{2}, p, \xi, \eta, (r - s)z; z, s, r, (p, r - s), \xi, \eta + pt, \xi - pt, \tau \right) \mid (z, s, r, p, \xi, \eta, \tau) \in \text{subset of } \mathbb{R}^{4n-1} \right\}
$$

$$
\subset T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0 \times T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0.
$$

(60)

Using the change of variables, $\xi = \tau + \rho$, $p = \frac{x}{2\tau}$, this canonical relation can be parameterized by the coordinates of $T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0$ except $t$, that is $(z, s, r, \xi, \sigma, \rho, \tau)$

$$
\left\{ \left( \frac{s + r}{2}, \frac{\tau}{2}, \xi, \sigma + \rho, (r - s)z; z, s, r, \left( \frac{\tau}{2}, r - s \right), \xi, \tau \right) \mid (z, s, r, \xi, \sigma, \rho, \tau) \in \text{subset of } \mathbb{R}^{4n-1} \right\}
$$

$$
\subset T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0 \times T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0.
$$

(61)

The projection of this canonical relation on $T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0$ is a hypersurface defined by

$$
t = \left( \frac{\tau}{2}, (r - s) \right).
$$

(62)

The map $d \mapsto L^*\psi, d$ is a Fourier integral operator with canonical relation

$$
\left\{ (z, \gamma; (0, \sigma, 0, \tau), t_0 + \gamma; (z, 0, \sigma, 0, \tau), \tau), t_0 + \gamma; (z, 0, \sigma, 0, \tau), \gamma; (z, 0, \sigma, 0, \tau), \gamma; (z, 0, \sigma, 0, \tau), \gamma; (z, 0, \sigma, 0, \tau), \gamma; (z, 0, \sigma, 0, \tau), \gamma; (z, 0, \sigma, 0, \tau), \gamma; (z, 0, \sigma, 0, \tau), (s_0, r_0, t_0, \sigma, 0, \rho, \tau) \mid (s_0, r_0, t_0, \sigma, 0, \rho, \tau) \in T^*\mathbb{R}^{2n-1} \setminus 0, z \in \mathbb{R}_+ \right\}
$$

$$
\subset T^*\mathbb{R}^{2n-1}_{(z,x,p)} \times T^*\mathbb{R}^{2n-1}_{(z,x,p)}
$$

(63)

derived from the canonical relation of $H(0, z)^*$, cf. (10). This canonical relation is parameterized by $(z, s_0, r_0, t_0, \sigma, 0, \rho, \tau)$, and is time translation invariant (in $t_0$). The line in the projection of this canonical relation on $T^*\mathbb{R}^{2n-1}_{(z,x,p)} \setminus 0$ parameterized by $t_0$ for fixed $(s_0, r_0, \sigma, 0, \rho, \tau)$ intersects the hypersurface (62) transversally. It follows that the composition of the canonical relations (63) and (60) is transversal. This composition is parameterized by $(z, s_0, r_0, \sigma, 0, \rho, \tau)$. It follows that $R_{\partial t} L^*$, and hence $A_{WE} = R_{\partial t} L^* D^2_{\tau}$, are Fourier integral operators. Let $p(z)$ denote the value of $\frac{\sigma^2}{2\tau}$ along a certain DSR bicharacteristic with initial values
of A composition is a Fourier integral operator with canonical relation
\[
A f = \int K(x,\xi) e^{i\langle x,\xi \rangle} \, d\xi
\]
for some \(\varphi, \Theta, \varphi\), and let \(t(z)\) denote the time and \(h(z)\) denote the value of \(r - s\). The elements of the canonical relation of \(A_{\text{WE}}\) correspond to solutions \((z, s_0, r_0, \sigma_0, \rho_0, \tau)\) of \(t(z) - \langle p(z), h(z) \rangle = 0\).

To analyze the composition \(A_{\text{WE}} F D_t^0\), we first analyze the composition \(L^*D_t^0\psi y LE_2 E_1\), that maps the perturbation \(\frac{1}{2}\zeta_0^{-3} \zeta c = (\frac{1}{2} \zeta_0^{-3} \zeta c)(z', x)\) to the downward continued data as a function of \((z, s, r, t)\). This composition is a Fourier integral operator with canonical relation
\[
\{(z, \gamma_z(z, z', x, \sigma', \tau), \eta_z(z, z', x, \sigma', \tau), \gamma_z(z, z', x, \sigma', \tau) + \gamma_z(z, z', x, \sigma', \tau), \zeta, \eta_z(z, z', x, \sigma', \tau), \gamma_z(z, z', x, \sigma', \tau), \tau; z', x, \sigma', \sigma', + \rho')\}
\]
where \((z', x, \sigma', \rho')\) is a subset of \(\mathbb{R}^{3n}\) such that \(\zeta = \Theta(z', x, x, \sigma', \tau, \zeta)\).

Here, the propagation of singularities upward by \(L\) and downward by \(L^*\) is along the same DSR bicharacteristics.

We now show that the composition \(A_{\text{WE}} D_t^2 LE_2 E_1\) is a Fourier integral with canonical relation contained in
\[
\{(z, x, p, \zeta, \xi, 0; z, x, \zeta, \xi) \in T^*\mathbb{R}^{2n}(z, x) \setminus 0, \|p\| < p_{\text{max}}\}.
\]
(65)

From that it follows that \(A_{\text{WE}} F D\) is a \(p\)-family of pseudodifferential operators. Indeed, the projection of (64) on \(T^*\mathbb{R}^{2n}(z, x, s, r, t)\) intersects the hypersurface (62) at \(z' = z\), since only then \(r - s = 0 = t\), leading to elements in (65), see Fig. 2. Since singularities propagate with speed less than \(C_0\), if \((1, v, v_0, v_1, v_{0x}, v_{0y}, 0, 0)\) derived from \(D^\tau_{\text{WE}}\) is a tangent vector to the DSR bicharacteristic \(\Gamma\) (cf. (10); the first component \(v_z = 1\), then \(h v_{\zeta} - v_{\zeta} t \leq C_0\), cf. (10)). By integrating this inequality we find that \(\|r - s\| \leq C_0\|t\|\) along \(\Gamma\). On the other hand, for points \((z, x, s, r, \zeta, \xi, 0, s, r, t, \sigma, \rho, \tau)\) in (60), one has that \(\|t\| = |\langle p, r - s \rangle| \leq p_{\text{max}}\|r - s\| < C_0^{-1}\|r - s\|\). From this, it follows that the composition of (64) with (60) contains no elements outside (65). It follows also that the composition of (64) with (60) is transversal.

Finally, we show that \(A_{\text{WE}}\) is invertible. To this end, we investigate the projections of the canonical relation of \(A_{\text{WE}}\) on the first component \(T^*\mathbb{R}^{2n}(z, x, s, r, t)\) \(0\) and on the second component \(T^*\mathbb{R}^{2n}(z, x, s, r, t)\) \(0\), respectively. The projection of the canonical relation of \(A_{\text{WE}}\) on the second component \(T^*\mathbb{R}^{2n}(z, x, s, r, t)\) \(0\) is invertible if each DSR bicharacteristic with initial values \((s_0, r_0, t_0, \sigma_0, \rho_0, \tau)\), parameterized by \(z\), intersects the hypersurface (62) at most once and transversally. We reconsider the equation, \(t(z) - \langle p(z), h(z) \rangle = 0\), defining the canonical relation of \(A_{\text{WE}}\). To estimate the derivative of the left-hand side, we will argue that
\[
\frac{\partial t}{\partial z} - \left\langle p(z), \frac{\partial h}{\partial z}(z) \right\rangle < -\epsilon_0
\]
for some \(\epsilon_0 > 0\) depending on \(p_{\text{max}}\) and \(R\) as in the theorem. Indeed, note that the factor
\[
1 - \left\langle p(z), \frac{\partial h}{\partial z}(z) \right\rangle \left(\frac{\partial t}{\partial z}\right)
\]
Fig. 2. Contribution to the kernel of \(A_{\text{WE}} D_t^2 LE_2 E_1\) (here, \(n = 2; h = r - s\) is offset).
is bounded away from zero. Also, note that $\frac{\partial \varphi}{\partial x}$ is strictly negative while $|\frac{\partial \varphi}{\partial x}| \geq \frac{1}{\epsilon_0}$. Since according to the Hamilton equations,

$$\frac{d\gamma_i}{dx} = \frac{\partial b}{\partial \gamma_i} = \frac{-\tau}{2\sqrt{c_0^{-2} - \tau^{-2}} \|\xi\|^2} \frac{\partial c_0^{-2}}{\partial x}.$$  \hfill (67)

it follows that $\|\frac{\partial \varphi}{\partial x}(z)\| \leq \frac{C_1}{\tau}$, so that $\|h\| < C_3 \frac{\cos(\theta_2)}{C_0 c_1 c_2}$, it follows that

$$\left|\frac{\partial \varphi}{\partial x}(z), h(z)\right| < C_3 \frac{1}{2C_0} < \epsilon_0$$  \hfill (68)

by an appropriate choice of $C_3$. This implies that the function $z \mapsto t(z) - \langle p(z), h(z) \rangle$ is monotone. Hence the projection of the canonical relation of $A_{WE}$ on $T^* \mathbb{R}^{2n-1}_{(x,\ell)} \setminus 0$ is invertible.

From canonical relation (60) we now establish that, given $z, x = x, \sigma, \rho, \tau$), the map $(h = x - r, \tau) \mapsto (\vartheta = (r - s)\tau, \zeta = \Theta(x, \sigma, r, \rho, \tau))$ is invertible for $h$ sufficiently small. Assume that $(h_1, \tau_1)$ maps to $(\vartheta_1, \zeta_1)$ and that $(h_2, \tau_2)$ maps to $(\vartheta_2, \zeta_2)$ with $\vartheta_2 = \vartheta_1$. We show that $\zeta_1 > \zeta_2$ if $\tau_1 > \tau_2$. Estimating the difference

$$\Theta(x, \frac{h_1}{2r} - \frac{h_2}{2r}, \sigma, \rho, \tau_1) - \Theta(x, \frac{h_1}{2r} - \frac{h_2}{2r}, \sigma, \rho, \tau_2),$$  \hfill (69)

using the bounds

$$\left|\frac{\partial b}{\partial \tau}\right| = \frac{c_0^{-2}}{\sqrt{c_0^{-2} - \tau^{-2} \|\xi\|^2}} \leq \frac{C_5}{C_4} \frac{\tau}{\|\xi\|^2},$$  \hfill (70)

in which $C_4$ depends on the lower and upper bounds of $c_0$, and $C_5 = \frac{1}{c_2}$ is a lower bound for $c_0^{-2}$, and (cf. (67))

$$\left|\frac{\partial b}{\partial \tau}\right| = \frac{c_0^{-2}}{\sqrt{c_0^{-2} - \tau^{-2} \|\xi\|^2}} \leq \frac{\tau}{\|\xi\|^2} \leq \frac{\tau}{C_6} \|\xi\|,$$  \hfill (71)

where $C_6$ depends on $\theta_2$ and the upper and lower bounds of $c_0$, yields the estimate

$$\zeta_1 - \zeta_2 \geq \frac{C_5(\tau_1 - \tau_2)}{C_4} \|\xi\| \|\tau_1| - \tau_2\|.$$  \hfill (72)

Since $\vartheta_2 = \vartheta_1, (h_1 - h_2)\tau_1 = h_2(\tau_2 - \tau_1)$, so that

$$\frac{C_5(\tau_1 - \tau_2)}{C_4} \geq \frac{\|\xi\|}{C_6} \|\tau_1| - \tau_2\| \leq \frac{C_5}{C_4} \|\tau_1 - \tau_2\| \frac{\|\xi\|}{C_6} \|\tau_2| - \tau_1\|.$$  \hfill (73)

The right-hand side of this equality is strictly greater than zero for $\|h_2\|$ sufficiently small, since $\tau_1 - \tau_2 > 0$. It follows from this that $\zeta_1 > \zeta_2$. We conclude that the projection of the canonical relation of $A_{WE}$ on $T^* \mathbb{R}^{2n-1}_{(x,\ell)} \setminus 0$ is invertible as well. It follows also that the linearization of this projection is invertible. This establishes the last statement of the theorem.  \hfill $\square$

To conclude this section we determine, at the principal symbol level, the modification of (56) that leads to reconstruction of singularities of $\delta c$ microlocally for each $p$ subject to $\|p\| < p_{\text{max}}$. Like the reconstruction in (47) of Theorem 2.2, the reconstruction is microlocal. The three cutoffs, $\chi$ and $\psi, \psi_D$, must be taken into account. The canonical relation of $A_{WE}$ defines a map $(s, r, t, \sigma, \rho, \tau) \mapsto (z, x, p, \xi, \vartheta)$ (where $\vartheta$ is the $p$-covector); there is also an associated value of $h = r - s$ through (60). By pull back with the inverse of the mentioned map, one can map the symbols $\psi, \psi_D$ to symbols in the variables $(z, x, p, \xi, \vartheta)$. By the mentioned evaluation of $h$ one obtains by pull back the cutoff $\chi$ in these variables also. We define $\Psi_{WE} = \Psi_{WE}(z, x, p, \xi, \vartheta)$ as the product of these symbols. With this definition we have
Proposition 3.2. Define $\tilde{A}_{WE}$ by
\[
(\tilde{A}_{WE})_d(z,x,p) = j^{-1} R_3 \bar{Z}^{-1} Q_{-\gamma}^{-1}(z)^{-1} Q_{-\gamma}(z)^{-1} L^* Q_{-\gamma}(0)^{-1} Q_{-\gamma}(0)^{-1} D^*_1 \psi_d,
\]
in which $\bar{Z} = \bar{Z}(z,s,r,\Theta(z,s,r,\sigma,\rho,\tau),\sigma,\rho)$ is independent of $t$ (the corresponding operator is convolutional in time), and where the symbol of $\gamma$ is defined in (86) and (87) below. Suppose Assumption 1 holds. Suppose that $\gamma$ is 1 on a neighborhood of $h = 0$ and $h \mapsto \gamma(z,x) = h$ is supported in $B(0, R)$ (cf. Theorem 3.1), then $\tilde{A}_{WE}$ is an invertible Fourier integral operator. Let the symbol $\Psi_{WE}$ be as defined above. The composition $A_{WE}F$ is a $p$-family of pseudodifferential operators with principal symbol $\Psi_{WE}(z, x, p, \zeta, \xi, \rho)$. The microlocal reconstruction hence follows from
\[
(\Psi_{WE}(z,x,p,D_z,D_x,0) + \text{order}(-1))(\frac{1}{h} \partial_h) = \tilde{A}_{WE} d.
\]
We note that $\Psi_{WE}$ accounts purely for illumination and cannot be compensated for.

**Proof.** We consider the operator $R_3 \bar{Z}^{-1} L^* E_2 E_1$ or the map
\[
(\frac{1}{h} \partial_h) \mapsto \int \left( \bar{Z}(z) \right)^{-1} H(0,z)^* \int H(0,z') E_2 E_1 \left( \frac{1}{h} \partial_h \right) (x - \frac{h}{2}, x + \frac{h}{2}, ph) dh,
\]
and evaluate, microlocally, its principal symbol. This principal symbol will be the principal symbol of operator $\gamma$ in the theorem. In this proof, we will omit the cutoff functions that are part of the symbols; the calculations will be valid microlocally on the support of a cutoff.

Using an oscillatory integral representation of $H$ similar to the one in the proof of Theorem 2.2, we find that the principal contribution to the kernel of this map, as a function of $(z, x, p, z', x')$, can be written as
\[
(2\pi)^{-2n} \int \bar{Z}(z,x - \frac{h}{2}, x + \frac{h}{2}, ph, y_0, \eta_0) A(z', x', 0, y_0, \eta_0) \exp[i(\theta(z, x, p, z', y_0, \eta_0)) \partial_y \partial_\eta dh,
\]
where $\frac{\partial S}{\partial z}$ are evaluated at $(z, x - \frac{h}{2}, x + \frac{h}{2}, ph, y_0, \eta_0)$, and where
\[
\Phi_R(z,x,p,z',x',y_0,\eta_0) = -S(z,x - \frac{h}{2}, x + \frac{h}{2}, ph, y_0, \eta_0) + S(z', x', 0, y_0, \eta_0).
\]
We expand this phase in a Taylor series about $(z', x', h) = (z, x, 0)$ and identify the gradient at $(z, x, 0)$,
\[
- \frac{\partial \Phi_R}{\partial z}(z,x,0,y_0,\eta_0) = \zeta(z,x,0,y_0,\eta_0),
\]
\[
- \frac{\partial \Phi_R}{\partial x}(z,x,0,y_0,\eta_0) = \sigma(z,x,0,y_0,\eta_0) + \rho(z,x,0,y_0,\eta_0),
\]
\[
- \frac{\partial \Phi_R}{\partial h}(z,x,0,y_0,\eta_0) = \frac{1}{2}\sigma(z,x,0,y_0,\eta_0) + \frac{1}{2}\rho(z,x,0,y_0,\eta_0) + \tau(z,x,0,y_0,\eta_0),
\]
where
\[
- \frac{\partial S}{\partial (s,r,t,z)}(z,x,0,y_0,\eta_0) = (\sigma(z,x,0,y_0,\eta_0), \rho(z,x,0,y_0,\eta_0), \tau(z,x,0,y_0,\eta_0), \zeta(z,x,0,y_0,\eta_0)).
\]
Applying a change of variables, $(y_0, \eta_0) \mapsto (\zeta, \sigma, \rho)$, the phase takes the form
\[
\zeta(z - z') + \langle \sigma + \rho, x - x' \rangle + \langle \frac{1}{2}(\rho - \sigma) + \tau, h \rangle, \quad \text{with} \quad \tau = \Theta^{-1}(z, x, z', \zeta, \sigma, \rho).
\]
The amplitude factor $\bar{Z}^{-1} A$ at $h = 0$ becomes equal to one by the calculations in the proof of Theorem 2.2. Upon changing integration variables, $\sigma = \frac{1}{2} \zeta - \vartheta, \rho = \frac{1}{2} \zeta + \vartheta$, the oscillatory integral (77) takes the leading-order form.
\[
(2\pi)^{-(2n-1)} \int e^{i\langle \zeta, z', \xi, x' + (\vartheta + tp, h) \rangle} d\zeta d\xi dh, \tau = \Theta^{-1}(z, x, x', \zeta, \frac{1}{2}z, -\vartheta, \frac{1}{2}z + \vartheta). \tag{84}
\]

By the method of stationary phase (see e.g. [9, section 1.2]) the following integral is a symbol
\[
(2\pi)^{-(n-1)} \int e^{i\langle \vartheta + \Theta^{-1}(z, x, x', \zeta, \frac{1}{2}z - \vartheta, \frac{1}{2}z + \vartheta, p, h) \rangle} d\vartheta dh; \tag{85}
\]
it follows from (84) that this equals to highest order the symbol of \(\tilde{A}_{WE}F\). By the method of stationary phase the principal part of this symbol is given by
\[
j(z, x, p, \zeta, \xi) := \left| \frac{\partial((\vartheta + \Theta^{-1}(p, h)))^{-\frac{1}{2}}}{\partial(h, \vartheta)} \right|, \tag{86}
\]
evaluated where \(h = 0\) and \(\vartheta\) is such that \(\vartheta + \Theta^{-1}p = 0\). By evaluating this, we find that
\[
\left| \frac{\partial((\vartheta + \Theta^{-1}(p, h)))^{-\frac{1}{2}}}{\partial(h, \vartheta)} \right| = \left| \frac{\partial(\vartheta + \Theta^{-1}(p))}{\partial \vartheta} \right| = \left| \det \left( I + p \otimes \left( \frac{\partial \Theta^{-1}}{\partial \vartheta} - \frac{\partial \Theta^{-1}}{\partial \rho} \right) \right) \right|, \tag{87}
\]
evaluated at \(s = r = x\) and \(\sigma = \frac{1}{2}z - \vartheta, \rho = \frac{1}{2}z + \vartheta\). This completes the proof. \(\square\)

**Remark 3.3.** Since \(\tilde{A}_{WE}\) is invertible, microlocally, we have obtained the following diagram as suggested by Symes [31]
\[
\begin{array}{ccc}
\mathcal{S}'(X \times E) & \xrightarrow{\tilde{A}_{WE}} & \mathcal{D}'(Y) \\
\Psi_{WE} \uparrow & & \uparrow \text{Id} \\
\mathcal{S}'(X) & \xrightarrow{F_0} & \mathcal{D}'(Y)
\end{array} \tag{88}
\]
where \(E = \{p \in \mathbb{R}^{n-1} \mid \|p\| < p_{\text{max}}\}\) as in Theorem 3.1 (cf. (58)). In this diagram – through the introduction of \(\mathcal{S}'(X \times E)\) – the redundancy in the data, measured by \(\dim Y - \dim X\), is manifest.

### 4. Annihilators

As discussed in the introduction, the inverse problem of determining the background medium \(c_0\) can be addressed by making use of the redundancy in the data. The background medium must be such that the data is in the range of the operator \(F\). The data are in the range of \(F_0\), if the angle transform generates a collection of identical images of \(\frac{1}{2}c_0^{3} \delta c\), parameterized by \(p = (p_1, \ldots, p_{n-1})\), microlocally. It follows that we have the following criterion
\[
(\tilde{A}_{WE}[c_0]d)(z, x, p) \text{ is independent of } p, \tag{89}
\]
microlocally, or
\[
\frac{\partial}{\partial p_i} (\tilde{A}_{WE}[c_0]d)(z, x, p) = 0, \tag{90}
\]
microlocally. Seismologists recognize this as “alignment” of the singularities in common-image-point gathers. Of course it must be taken into account that \(\tilde{A}_{WE}\) is only a microlocal inverse, so (90) is not valid globally.

The criterion (90) can be restated as that certain pseudodifferential operators annihilate the singular part of the data, see [26]. With the approach based on inversion for \(\delta c\) using subsets of data (discussed in the introduction) this is closely related to differential semblance [30]. A construction of such operators, annihilators, follows straightforwardly from the angle transform introduced in the previous section. On transformed data \(\tilde{A}_{WE}d\), i.e. on common-image-point gathers, annihilators are given by \(\frac{\partial}{\partial p_i}\, , i = 1, \ldots, n - 1\), cf. (90). Indeed, using Proposition 3.2, it follows that where there is illumination, \(\frac{\partial}{\partial p_i} \tilde{A}_{WE}d = 0\) (cf. (75)). Annihilators of the data then follow to be
\[
\left( \frac{\partial}{\partial t} \right)^{-1} \langle \tilde{A}_{\text{WE}} \rangle^{-1} \frac{\partial}{\partial p_i} \tilde{A}_{\text{WE}}
\]  
(91)

as pseudodifferential operators of order zero, where \( \langle \tilde{A}_{\text{WE}} \rangle^{-1} \) is a regularized inverse for \( \tilde{A}_{\text{WE}} \), that is supported microlocally on a subset of \( T_{C_{0}^{p},p}^{\infty} \) where \( A_{\text{WE}} \) is invertible. Such annihilators are not uniquely defined. Indeed (91) can be composed on the left by any invertible order 0 pseudodifferential operator and we still have an annihilator, of the same order.

We now derive an alternative definition of an annihilator. This definition is motivated by the observation that

\[
\frac{\partial}{\partial p_i} R_3 \left( \frac{\partial}{\partial t} \right)^{-1} g = \int_{p=0} M_j (R_2 g) (z, x - \frac{h}{2}, x + \frac{h}{2}) \chi(z, x, h) dh,
\]

where \( M_j \) denotes the multiplication by \( h_j = (r_j - s_j) \), and \( \chi \) was introduced in (56). We have \( M_j E_1 = 0 \), since \((r_j - s_j)\delta(r - s) = 0 \). We observe first that using Theorem 4.2 of Paper I,

\[
M_j K^* D^2_i d = K^* D^2_i KM_j E_1 \left( \frac{1}{2} \frac{\delta c}{\delta x} \right) + [M_j, K^* D^4_i K] E_1 \left( \frac{1}{2} \frac{\delta c}{\delta x} \right).
\]

(92)

The first term vanishes (because \( M_j E_1 = 0 \)), and the second term is of lower order, so \( M_j \) is to highest order an annihilator for \( K^* D^2_i d \).

To derive from (92) a pseudodifferential annihilator of the data, \( M_j K^* D^2_i d \) is multiplied on the left by \( D^2_i K \). We must also have the support of the operator on the left smaller than the support of the operator on the right of \( M_j \). So suppose \( \psi_j \) is in \( C_0^p (Y) \) and is 1 on \( \text{supp} (\psi_j) \), and \( K^* \) is defined as \( K \) but with pseudodifferential cutoff at larger angles of propagation \( \theta_1, \theta_2 \) as introduced above (31). Define

\[
W_j = \psi_j D^2_i K M_j (K^* D^2_i \psi_j D^2_i K')^{-1} K^* D^2_i \psi_j,
\]

(93)

where we recognize \( N_{K^*} := K^* D^2_i \psi_j D^2_i K' \) from Lemma 2.1. Then we have

Theorem 4.1. With Assumption 1, the \( W_j \) and the operators (91) are annihilators of the data.

We note that annihilators \( W_j \) are essentially multiplicative and do not require a differentiation as in (91), and hence are the ones to be used in practical implementations.

We could define \( \tilde{K} = KN_{K}^{-1/2} \) as a modification of \( K \) so that \( D^2_i \tilde{K} \) is unitary – at least where \( \psi_D \) is 1. Then \( \frac{1}{2} \sum_j ||W_j||_d \|^2 \) simplifies approximately to (the norm does not change by removing \( D^2_i \tilde{K} \) from the annihilators)

\[
\frac{1}{2} \sum_j ||W_j||_d \|^2 \approx \frac{1}{2} \int (||r - s||^2 ||\tilde{K}^* D^2_i d||^2 + \text{lower order terms}) dz ds dr,
\]

(94)

This expression is small when the time-to-depth converted data \( \tilde{K}^* D^2_i d \) are “focused” at \( r = s \).

Migration velocity analysis is the estimation of \( c_0 \) (the background or “velocity model”) based upon the alignment expressed by (90). Traditionally, the updating of the velocity model is carried out interactively. The problem of updating \( c_0 \) so that the data will be contained in the range of \( F_D \) is now cast in a minimum norm optimization problem. In practice, such problems can be addressed by, for example, the conjugate gradient method. Liu and Bleistein [15] developed an automated method for updating \( c_0 \) on the basis of the curvature of misalignment at \( p = 0 \) using ray perturbation theory. It was also done using the data subset based annihilators (differential semblance [30]). Annihilators, \( W_j = W_j[c_0] \), replace the necessity to estimate this curvature from the \( p \)-family of images. They depend on the background medium. The semi-norm \((\sum_j ||W_j||_d \|^2)^{1/2} \) detects whether \( c_0 \) was an acceptable choice or not. The functional \( \frac{1}{2} \sum_j ||W_j||_d \|^2 \) can be viewed as the downward continuation analog of the differential semblance functional of Symes [30] with the advantage that our annihilator admits the formation of caustics. Of course, the same is true with the \( W_j \) replaced by the operators in (91).

References