# Extended isochron rays in prestack depth (map) migration 

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#### Abstract

Many processes in seismic data analysis and seismic imaging can be identified with solution operators of evolution equations. These include data downward continuation and velocity continuation. We have addressed the question of whether isochrons defined by imaging operators can be identified with wavefronts of solutions to an evolution equation. Rays associated with this equation then would provide a natural way of implementing prestack map migration. Assuming absence of caustics, we have developed constructive proof of the existence of a Hamiltonian describing propagation of isochrons in the context of common-offset depth migration. In the presence of caustics, one should recast to a sinking-survey migration framework. By manipulating the double-square-root operator, we obtain an evolution equation that describes sinking-survey migration as a propagation in twoway time with surface data being a source function. This formulation can be viewed as an extension of the exploding reflector concept from zero-offset to sinking-survey migration. The corresponding Hamiltonian describes propagation of extended isochrons (fronts with constant two-way time) connected by extended isochron rays. The term extended reflects the fact that two-way time propagation now takes place in high-dimensional space with the following coordinates: subsurface midpoint, subsurface offset, and depth. Extended isochron rays can be used in a natural manner for implementing sinking-survey migration in a map-migration fashion.


## INTRODUCTION

Many processes in seismic data analysis and seismic imaging can be identified with solution operators (propagators) of evolution equations. These include data downward continuation (Clayton, 1978; Claerbout, 1985; Biondi 2006a) and velocity continuation (Fomel, 1994, 2003b; Goldin, 1994; Hubral et al., 1996b; Adler,
2002), where continuation relates to the evolution parameter in such equations. (For a theory of continuation in seismic imaging, see also Duchkov et al. [2008] and Duchkov and de Hoop [2008]). In this paper, we address the question of whether isochrons derived from imaging operators can be identified with wavefronts of solutions to an evolution equation.

From the physics point of view, evolution equations describe continuation (propagation) of energy in the direction of an increasing evolution parameter (time being the most common example). Fronts are defined as equal-phase surfaces and reveal the kinematics or propagation of a signal. Mathematically, propagation of fronts is described by a zero-level set of a Hamiltonian that follows from the symbol of the evolution equation (see Courant and Hilbert, 1991). Then rays can be introduced as bicharacteristics and constructed as solutions to corresponding Hamilton equations. In seismology, a Hamiltonian is usually associated with an eikonal equation and Hamilton equations appear to be a ray-tracing system. It should be noted, however, that ray-based techniques can be applied not only to the wave equation (Červený, 2001; Popov, 2002) but to all evolution equations that appear in seismic processing, e.g., for the DMO equation (Fomel, 2003a).

The role of continuation is clear in zero-offset ( ZO ) depth migration. Based on the exploding reflector concept, one can formulate ZO migration in terms of solving a wave equation (evolution in time) with half-velocities assuming the absence of caustics (Lowenthal et al., 1976; Cheng and Coen, 1984; Claerbout, 1985). The role of continuation is less obvious for prestack common-offset (CO) depth migration. The geometry of CO migration can be understood in terms of isochrons, which we will refer to as CO isochrons. A CO isochron is a hypersurface connecting all points in the subsurface that have the same combined traveltime along two rays connecting these subsurface points to a fixed source-receiver pair (source-receiver ray pair). CO isochrons are usually viewed as impulse responses of a CO migration operator (see Goldin, 1994, 1998; Hubral et al., 1996a). They form the basis for a so-called Kirchhoff-type migration that maps each point form data to an isochron surface contributing to an image (see, e.g., Bleistein et al., 2000).

As opposed to Kirchhoff-type migration methods that do not

[^0]make explicit use of the slopes in the data, the term map migration can be used for methods that aim to implement one-to-one mapping of seismic reflections (traveltimes and slopes in the data) to reflectors (positions and local dips in the image). Just to mention a few methods from this group: controlled directional reception (Zavalishin, 1981; Sword, 1987), parsimonious migration (Hua and McMechan, 2003), stereotomography (Billette and Lambaré, 1998), curvelet-based map migration (Douma and de Hoop, 2007), etc. In all of these methods, a preprocessing step is required for estimating slopes in the data using slant-stack analysis methods (Riabinkin, 1991) or curvelet/wave-packet decomposition (Candès et al., 2006; Andersson et al., 2008).

Given slopes in the data map migration seem to be computationally superior to Kirchhoff migration because data map migration avoids costly procedures of redundant summation/smearing. However, it also requires knowledge of mapping of reflections into reflectors. Evolution equations in seismic imaging and associated rays provide a natural way of implementing this mapping, i.e., connecting reflections to reflectors via continuation. For an illustration, one can consider a standard wave equation. A Kirchhoff-type approach to wavefront construction would be based on exploiting Huygens' principle. Rays can be used for constructing successive wavefronts in a more efficient manner (Lambaré et al., 1996). A related approach to describing wave propagation itself is based on moving Gaussian packets and beams along these rays (Babich and Ulin, 1984; Klimeš, 1989; Popov, 2002). Evolution equations discussed in this paper open the door to using similar procedures for implementing prestack depth migration.

CO isochrons are typically constructed using pairs of rays (socalled source and receiver rays), each of which is obtained by solving a separate system of Hamilton equations obtained from the wave equation. Because these isochrons have the appearance of fronts, the natural question to arise is whether they can be connected themselves with rays generated by a single Hamiltonian. Iversen (2004, 2005) connected them by curves that we refer to as CO isochron rays. In the construction of the CO isochron rays, Iversen (2004, 2005) also uses pairs of source and receiver rays assuming that there is an unknown single Hamiltonian but not proving its existence. In this paper, we show how to construct this Hamiltonian under the absence of caustics assumption.

In this paper, we consider prestack migration in the downwardcontinuation approach (see Claerbout, 1985; Stolk and de Hoop, 2001,2006 ) that is also called survey-sinking migration, one-way wave-equation migration, or double square-root (DSR) migration. The assumption used in this case is that the DSR condition is satisfied (Stolk and de Hoop, 2005): rays are nowhere horizontal (caustics are allowed). In this case, one can use the double square-root equation (Belonosova and Alekseev, 1967; Clayton, 1978; Claerbout, 1985), an evolution equation in depth, for the downward continuation of data. The DSR equation describes propagation of data in a $2 n$-dimensional DSR imaging volume with the following coordinates: subsurface midpoint, subsurface offset, two-way time, and depth (for 3D seismics, $n=3$ ). This equation provides us with a DSR Hamiltonian and an associated system of Hamilton equations for tracing DSR rays in this volume. A zero two-way time section in this volume will correspond with a prestack DSR image that depends on subsurface horizontal coordinates, subsurface offset, and depth (setting two-way time equal to zero becomes the DSR imaging condition). Note that a conventional image corresponds to a zero subsur-face-offset section of a prestack DSR image. Information at nonzero
subsurface offsets is important and can be used in migration velocity analysis (Shen et al., 2003; Shen, 2005).

The main goal of this paper is to derive Hamiltonians describing propagation of isochrons as fronts. In Appendix A, we construct such a Hamiltonian for CO migration. The corresponding rays appear to be isochron rays by Iversen (2004) in the combined parametrization. The construction in Appendix A also provides the limitations of the approach: We can obtain a Hamiltonian for CO isochron rays (see equation A-8) only in the absence of caustics. In the presence of caustics, it was observed by Iversen (2004) that CO isochron rays can branch, with two-way time not increasing monotonically along the ray. On the contrary, evolution parameters (two-way time in our case) have to grow monotonically along rays which are solutions to a Hamilton system.

We then consider generic velocity models that become possible in the framework of DSR migration. Kinematics of the DSR migration are described in details by Stolk et al. (2009). A short summary on DSR Hamiltonian and DSR rays is given in the next section, as well as a more detailed description of the DSR imaging volume.

Then we rewrite the DSR equation in the form of an evolution equation in two-way time (instead of depth). DSR migration is then formulated as an initial value problem with the data generating a source in this evolution equation. DSR demigration in this case can be viewed as an extension of the exploding reflector concept from poststack to DSR prestack depth migration (in Appendix B, we revisit the exploding reflector concept in more details). This Hamiltonian describes propagation of surfaces that can be now considered as fronts propagating from a point source (fixed midpoint, offset, and two-way time). In analogy with common-offset cases, we will call them multisubsurface-offset isochrons ( $M_{s} \mathrm{O}$ isochrons). They generalize a notion of isochrons being defined in the higher-dimensional DSR imaging volume. We refer to rays connecting these fronts as $M_{s} O$ isochron rays.

Then we provide an interpretation of kinematics of the DSR migration using $\mathrm{M}_{5} \mathrm{O}$ isochrons and $\mathrm{M}_{5} \mathrm{O}$ isochron rays. $\mathrm{M}_{5} \mathrm{O}$ isochrons are impulse-response surfaces for prestack DSR migration. Thus, they can be used for a smearing-type implementation of the prestack DSR migration. Note that $\mathrm{M}_{s} \mathrm{O}$ isochron rays provide a new parametrization for known DSR rays. DSR map migration can be implemented by tracing $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays for every detected event (traveltime and slopes) in data until zero two-way time to generate a DSR map-migrated image.

Later in the paper we discuss connections between CO and $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons and isochron rays. Then we show a few applications and examples illustrating a geometric part of DSR map migration.

## DSR RAYS AND DOWNWARD DATA CONTINUATION

Here, we summarize the approach to DSR migration based on downward data continuation using the double square-root equation. Throughout, we assume that the DSR condition holds (Stolk and de Hoop, 2005): For all source-receiver combinations present in the acquisition geometry, the source rays (each connecting a scattering or image point to a source) and receiver rays (each connecting a scattering or image point to a receiver) do not become horizontal (see Figure 1a). We note that the DSR condition can be generalized to hold with respect to a curvilinear coordinate system defining a pseudodepth, see Stolk et al. (2009)

## Data continuation in depth and DSR rays

In this subsection, we discuss the geometry of the operator of data continuation in depth. For downward data continuation, one commonly uses the double square-root equation (Belonosova and Alekseev, 1967; Clayton, 1978; Claerbout, 1985):

$$
\begin{equation*}
\left[\partial_{z}-\mathrm{i} P^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{D}_{s}, \mathbf{D}_{r}, D_{t}\right)\right] u=0 \tag{1}
\end{equation*}
$$

where $\partial / \partial z$ is depth; $\mathbf{x}_{s}$ is the horizontal source coordinate at fixed depth; $\mathbf{x}_{r}$ is the horizontal receiver coordinate at fixed depth; $D_{t}$ corresponds (in the Fourier domain) with multiplication by frequency $V ; \mathbf{D}_{r}$ corresponds (in the Fourier domain) with multiplication by wavevector $\mathbf{k}_{r}$; and $\mathbf{D}_{s}$ corresponds (in the Fourier domain) with multiplication by wavevector $\mathbf{k}_{s}$.

The DSR operator can be viewed, locally, as a pseudodifferential operator. Its principal symbol, denoted here by a subscript 1 , determines the associated propagation of singularities and ray geometry, that is, the kinematics. To guarantee dynamically correct propagation, one also needs the so-called subprincipal part of the symbol (de Hoop, 1996; Stolk and de Hoop, 2005). We suppress this contribution in the presentation here and focus on the geometry. The principal symbol of the DSR operator is given by the standard expression

$$
\begin{align*}
P_{1}^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{k}_{s}, \mathbf{k}_{r}, v\right)= & v \sqrt{\frac{1}{c\left(z, \mathbf{x}_{s}\right)^{2}}-\frac{\left\|\mathbf{k}_{s}\right\|^{2}}{v^{2}}} \\
& +v \sqrt{\frac{1}{c\left(z, \mathbf{x}_{r}\right)^{2}}-\frac{\left\|\mathbf{k}_{r}\right\|^{2}}{v^{2}}} \tag{2}
\end{align*}
$$

while considering $v .0 ; z \mathrm{P}[0, Z]$ denotes depth, where $Z$ denotes the maximum depth considered; $c(z, \mathbf{x})$ is a wave speed. For simplicity, we will consider an isotropic medium and waves of only one type, i.e., PP or SS reflected waves.

The propagation of singularities by solutions of equation 1 is governed by the Hamiltonian

$$
\begin{equation*}
\mathrm{H}^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, k_{z}, \mathbf{k}_{s}, \mathbf{k}_{r}, v\right)=k_{z}-P_{1}^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{k}_{s}, \mathbf{k}_{r}, v\right) \tag{3}
\end{equation*}
$$

resolved explicitly with respect to the vertical wavenumber, (see equation 2 ). We denote the DSR rays by


Figure 1. Geometry of the DSR operator for up/downward data continuation. (a) DSR ray viewed as a pair of conventional rays connecting surface source and receiver points $\left(\mathbf{x}_{0 s}, \mathbf{x}_{0 r}\right)$ to the subsurface source and receiver points $\left(\mathbf{x}_{s}, \mathbf{x}_{r}\right)$ at depth $z$ (note that these rays do not necessarily have to intersect at any depth); (b) DSR ray viewed as a solution (see equation 4) to the DSR Hamilton equations (see equation 5), with $\mathbf{X}$ defining position on a ray and $\mathbf{K}$ defining its orientation.

$$
\begin{align*}
& \mathbf{X}(z)=\left(\mathbf{x}_{s}\left(z ; z_{0}, \boldsymbol{G}_{0}\right), \mathbf{x}_{r}\left(z ; z_{0}, \mathcal{G}_{0}\right), t\left(z ; z_{0}, \mathcal{G}_{0}\right)\right), \\
& \mathbf{K}(z)=\left(\mathbf{k}_{s}\left(z ; z_{0}, \boldsymbol{G}_{0}\right), \mathbf{k}_{r}\left(z ; z_{0}, \mathcal{G}_{0}\right), \boldsymbol{v}\left(z ; z_{0}, \boldsymbol{G}_{0}\right)\right), \tag{4}
\end{align*}
$$

which solve the Hamilton system

$$
\begin{equation*}
\frac{\mathrm{d}\left(\mathbf{x}_{s}, \mathbf{x}_{r}, t\right)}{\mathrm{d} z}=\frac{\partial \mathrm{H}^{\mathrm{DSR}}}{\partial\left(\mathbf{k}_{s}, \mathbf{k}_{r}, v\right)}, \quad \frac{\mathrm{d}\left(\mathbf{k}_{s}, \mathbf{k}_{r}, v\right)}{\mathrm{d} z}=-\frac{\partial \mathrm{H}^{\mathrm{DSR}}}{\partial\left(\mathbf{x}_{s}, \mathbf{x}_{r}, t\right)}, \tag{5}
\end{equation*}
$$

for initial conditions $G_{0}=\left(\mathbf{x}_{s 0}, \mathbf{x}_{r 0}, t_{0}, \mathbf{k}_{s 0}, \mathbf{k}_{r 0}, \boldsymbol{V}_{0}\right)$, the starting point and orientation of the ray. Here, depth is the evolution parameter. Slowness vectors are obtained from the wavevectors through $\mathbf{k} / v$.

The solution to the system of equations 5 will be a curve $(\mathbf{X}(z), \mathbf{K}(z))$ in phase space with coordinates $(\mathbf{X}, \mathbf{K})$. The term ray is commonly associated with a curve $\mathbf{X}(z)$ in physical space with coordinates $\mathbf{X}$. The $\mathbf{K}$ coordinates are frequency-scaled slowness vectors associated with points along the ray $\mathbf{X}(z)$.

In Figure 1a, we show a typical representation of the geometry as a couple of rays in two copies of physical space, viz., one with coordinates $\left(z, \mathbf{x}_{s}\right)$ and one with coordinates $\left(z, \mathbf{x}_{r}\right)$. In Figure 1 b , we show a DSR ray as a single curve defined by equation 4 in the DSR imaging volume with coordinates $\left(\mathbf{x}_{s}, \mathbf{x}_{r}, z, t\right)$. One can see that the DSR ray (Figure 1b) does not require that two corresponding conventional rays (Figure 1a) start from the same subsurface point. These nonphysical ray combinations are a specific feature of the DSR migration kinematics (see Stolk et al., 2009).

## Imaging via continuation in depth (DSR)

In this subsection, we summarize results pertaining to seismic data modeling and imaging, subject to the single scattering approximation, in terms of solving Cauchy initial value problems in depth.

DSR equation 1 is usually interpreted as an equation for upward/ downward continuation of data, i.e. it allows us to recalculate data as if it were recorded at different depth as schematically illustrated in Figure 2 a (for the depth interval above the region containing scatterers). A more interesting result is that seismic reflection data can be modeled, in the Born approximation, with an inhomogeneous DSR equation (Stolk and de Hoop, 2005; Stolk et al., 2009, after coordinate transformation defined by equations 11):


Figure 2. Upward/downward data continuation. (a) Solutions to initial value problems for the homogeneous DSR equation 1 allows recalculating data from the surface to a data at given depth (sinkingsurvey concept); (b) Reflection data modeling can be made by solving initial value problem 6 with the inhomogeneous DSR equation (extended reflectivity becomes a source function).

$$
\begin{gather*}
(-)\left[\partial_{z}-\mathrm{i} P^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{D}_{s}, \mathbf{D}_{r}, D_{t}\right)\right] u=d(t) E\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}\right), \\
\left.u\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, t\right)\right|_{z=Z}=0, \tag{6}
\end{gather*}
$$

where

$$
\begin{equation*}
E\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}\right)=d\left(\mathbf{x}_{r}-\mathbf{x}_{s}\right) \frac{d c}{2 c^{3}}\left(z, \frac{1}{2}\left(\mathbf{x}_{r}+\mathbf{x}_{s}\right)\right) \tag{7}
\end{equation*}
$$

and $d c(z, \mathbf{x})$ is the function containing the rapid velocity variations representative of the scatterers, superimposed on a smooth background model described by the function $c(z, \mathbf{x})$. Equation 6 is solved in the direction of decreasing $z$ (upward and forward in time). The data are then modeled by the solution of equation 6 restricted to $z$ $=0: u\left(z=0, \mathbf{x}_{s}, \mathbf{x}_{r}, t\right)$. The corresponding solution operator is schematically illustrated in Figure 2b.

Imaging reflection data, $d=d\left(\mathbf{x}_{s}, \mathbf{x}_{r}, t\right)$, can be formulated as solving the initial value problem for an adjoint to the DSR equation,

$$
\begin{gather*}
{\left[\partial_{z}-\mathrm{i} P^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{D}_{s}, \mathbf{D}_{r}, D_{t}\right)\right] \quad u=0,} \\
\left.u\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, t\right)\right|_{z=0}=d\left(\mathbf{x}_{s}, \mathbf{x}_{r}, t\right), \tag{8}
\end{gather*}
$$

now to be solved in the direction of increasing $z$ (downward and backward in time). A conventional image at $(z, \mathbf{x})$ is obtained upon subjecting the solution of initial value problem defined by equation 8 to the imaging conditions: $u\left(z, \mathbf{x}_{s}=\mathbf{x}, \mathbf{x}_{r}=\mathbf{x}, t=0\right)$. A prestack DSR image at $\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}\right)$ is obtained applying only one of the imaging conditions: $u\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, t=0\right)$.

Through the above formulation, imaging is described to act in the prestack DSR imaging volume, as a composition of a continuation operator (in depth) and restriction operators (imaging conditions).

## MULTISUBSURFACE-OFFSET ISOCHRON RAYS AND PRESTACK IMAGE PROPAGATION

In Appendix A, we derive a Hamiltonian describing propagation of CO isochrons assuming the absence of caustics. The corresponding rays appear to be isochron rays by Iversen (2004) in the combined parametrization. In the presence of caustics, it was observed (see Iversen, 2004) that CO isochron rays can branch, with two-way time not changing monotonically. This is not surprising, taking into account that CO migration generates artifacts in the presence of caustics. However, we can still use the same approach while using extended imaging operators, i.e., DSR prestack depth migration, which is elaborated in this section.

Based on the DSR approach to imaging, we formulate the same procedure as a prestack image continuation in two-way time (the TWT approach). As a result, we rewrite DSR rays in the form of mul-tisubsurface-offset $\left(\mathrm{M}_{\mathrm{s}} \mathrm{O}\right)$ isochron rays with two-way time being an evolution parameter. We then formulate modeling and imaging procedures as solutions to Cauchy initial value problems for an evolution equation in two-way time.

## Prestack image propagation, $\mathrm{M}_{s} \mathrm{O}$ isochron rays

The DSR condition implies that any DSR ray (see Figure 1b) cannot turn in the $z$, or in the $t$, direction. Thus, two-way time $t$ in equation 4 is a monotonous function in $z$. This in turn implies that each DSR ray intersects planes $z=$ const and $t=$ const not more than one time. Thus, any of these planes can be chosen to pose an initial value
problem for data continuation, and either $z$ or $t$ can be used as an evolution parameter. Note that this corresponds to just another parametrization along the same DSR ray.

To begin with, let, for given ( $\left.z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{k}_{s}, \mathbf{k}_{r}\right), Q$ denote the mapping $v!k_{z}=P_{1}^{\mathrm{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{k}_{s}, \mathbf{k}_{r}, v\right)$ (see equation 3 ). Under the DSR condition there exists an inverse mapping, $Q^{-1}: k_{z}!v=$ $Q^{-1}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, k_{z}, \mathbf{k}_{s}, \mathbf{k}_{r}\right)$, that solves the equation (Stolk and de Hoop, 2005, Lemma 4.1):

$$
\begin{equation*}
k_{z}=P_{1}^{\operatorname{DSR}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, \mathbf{k}_{s}, \mathbf{k}_{r}, Q^{-1}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, k_{z}, \mathbf{k}_{s}, \mathbf{k}_{r}\right)\right) \tag{9}
\end{equation*}
$$

Then we get

$$
\begin{align*}
v & =Q^{-1}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, k_{z}, \mathbf{k}_{s}, \mathbf{k}_{r}\right) \\
& =\frac{c_{r} c_{s}}{\left|c_{-}^{2}\right|} k_{z} \sqrt{c_{+}^{2}+k_{z}^{-2}\left(\left\|\mathbf{k}_{r}\right\|^{2}-\left\|\mathbf{k}_{s}\right\|^{2}\right) c_{-}^{2}-2 \sqrt{c_{s}^{2} c_{r}^{2}+k_{z}^{-2}\left(\left\|\mathbf{k}_{r}\right\|^{2} c_{r}^{2}-\left\|\mathbf{k}_{s}\right\|^{2} c_{s}^{2}\right) c_{-}^{2}},} \tag{10}
\end{align*}
$$

where $c_{r}=c\left(z, \mathbf{x}_{r}\right), c_{s}=c\left(z, \mathbf{x}_{s}\right), c_{-}^{2}=c_{s}^{2}-c_{r}^{2}$, and $c_{+}^{2}=c_{s}^{2}+c_{r}^{2}$. We note that equation 10 allows the limit $c_{-}^{2}$ ! 0 to be taken.

It is often useful to transform coordinates from subsurface lateral source and receiver coordinates to subsurface midpoint and offset coordinates,

$$
\begin{gather*}
\mathbf{x}_{r}=\mathbf{x}+\mathbf{h}, \quad \mathbf{x}_{s}=\mathbf{x}-\mathbf{h},  \tag{11}\\
\mathbf{k}_{s}=\frac{1}{2}\left(\mathbf{k}_{x}-\mathbf{k}_{h}\right), \quad \mathbf{k}_{r}=\frac{1}{2}\left(\mathbf{k}_{x}+\mathbf{k}_{h}\right) .
\end{gather*}
$$

After this coordinate transformation, we get for equation 10

$$
\begin{align*}
v= & \frac{c_{r} c_{s}}{\left|c_{-}^{2}\right|} k_{z} \\
& 3 \sqrt{c_{+}^{2}+\frac{\left\langle\mathbf{k}_{x}, \mathbf{k}_{h}\right\rangle}{k_{z}^{2}} c_{-}^{2}-\sqrt{4 c_{s}^{2} c_{r}^{2}-\frac{\left(\left|\mathbf{k}_{x}\left\|^{2}+\right\| \mathbf{k}_{h}\right|^{2}\right)}{k_{z}^{2}}\left(c_{-}^{2}\right)^{2}+2 \frac{\left\langle\mathbf{k}_{x} \mathbf{k}_{h}\right\rangle}{k_{z}^{2}} c_{+}^{2} c_{-}^{2}}}, \tag{12}
\end{align*}
$$

where, now, $c_{s}=c(z, \mathbf{x}-\mathbf{h})$ and $c_{r}=c(z, \mathbf{x}+\mathbf{h})$.
Equation 12 can be interpreted as a characteristic equation $\mathrm{H}^{\text {TWT }}$ $=0$ for the Hamiltonian

$$
\begin{equation*}
\mathrm{H}^{\mathrm{TWT}}\left(z, \mathbf{x}, \mathbf{h}, v, k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right)=v-Q^{-1}\left(z, \mathbf{x}, \mathbf{h}, k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right), \tag{13}
\end{equation*}
$$

that is equivalent to equation 3 but explicitly resolved with respect to frequency. In case of a vertically inhomogeneous medium, $c(z, \mathbf{x})$ $=c(z)$ expressions for $\mathrm{H}^{\mathrm{TWT}}$ take a more simple form, as shown in Appendix B.

The Hamiltonian defined by equation 13 is describing the propagation of fronts (propagation of singularities) in $(z, \mathbf{x}, \mathbf{h})$. We will call these fronts multisubsurface-offset $\left(\mathrm{M}_{\mathrm{s}} \mathrm{O}\right)$ isochrons because they correspond to constant two-way time. In terms of conventional rays, $\mathrm{a}_{s} \mathrm{O}$ isochron is composed of all points $(\mathbf{x}, \mathbf{h})$ in the subsurface that provide the same two-way traveltime when connected to the surface source and receiver pair (see Figure 1). Note that these rays will start from separated points when $\mathbf{h} \neq 0$.

We note that $\mathrm{H}^{\mathrm{TWT}}$ is anisotropic even though the underlying wave equation has been taken in an isotropic medium. A constant velocity case is illustrated in Figure 3. In Figure 3a, we show a slowness surface defined by the equation $\mathrm{H}^{\mathrm{TwT}}=0$, whereas in Figure 3 b we show corresponding group velocity surface. We note that in the
constant velocity case, the group velocity surface is an example of an $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron.

We denote $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays by

$$
\begin{gather*}
\tilde{\mathbf{X}}(t)=\left(z\left(t ; t_{0}, \widetilde{G}_{0}\right), \mathbf{x}\left(t ; t_{0}, \widetilde{G}_{0}\right), \mathbf{h}\left(t ; t_{0}, \widetilde{G}_{0}\right)\right), \\
\tilde{\mathbf{K}}(t)=\left(k_{z}\left(t ; t_{0}, \widetilde{G}_{0}\right), \mathbf{k}_{x}\left(t ; t_{0}, \widetilde{G}_{0}\right), \mathbf{k}_{h}\left(t ; t_{0}, \widetilde{G}_{0}\right)\right), \tag{14}
\end{gather*}
$$

which solve the Hamilton system

$$
\begin{equation*}
\frac{\mathrm{d}(z, \mathbf{x}, \mathbf{h})}{\mathrm{d} t}=\frac{\partial \mathrm{H}^{\mathrm{TWT}}}{\partial\left(k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right)}, \quad \frac{\mathrm{d}\left(k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right)}{\mathrm{d} t}=-\frac{\partial \mathrm{H}^{\mathrm{TWT}}}{\partial(z, \mathbf{x}, \mathbf{h})}, \tag{15}
\end{equation*}
$$

for initial conditions $\widetilde{G}_{0}=\left(\widetilde{\mathbf{X}}_{0}, \widetilde{\mathbf{K}}_{0}\right)=\left(z_{0}, \mathbf{x}_{0}, \mathbf{h}_{0}, k_{z 0}, \mathbf{k}_{x 0}, \mathbf{k}_{h 0}\right)$ at initial time $t_{0}$. These initial conditions define the starting point and orientation of the isochron ray (they also imply the starting points and orientations of the pair of corresponding conventional source-receiver rays).

Note that the $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron ray defined by a system of equations 14 describes the same curve as the DSR ray 4 (see also Figure 1). Two-way time has become the evolution parameter instead of depth. Also, ray tracing systems described by equations 5 and equations 15 are different. Choosing two-way time as an evolution parameter implies that one can apply wavefront construction methods to the Hamiltonian system 15 to generate the smearing surface per trace in the data.

We can associate the Hamiltonian defined by equation 13 with an evolution
$\left[\partial_{t}-\mathrm{i} P^{\mathrm{TWT}}\left(z, \mathbf{x}, \mathbf{h}, D_{z}, \mathbf{D}_{x}, \mathbf{D}_{h}\right)\right] \widetilde{u}=0$,
where, now, the evolution parameter is two-way time $t$ and the related principal symbol is given by (see equation 12)

$$
\begin{align*}
& P_{1}^{\mathrm{TWT}}\left(z, \mathbf{x}, \mathbf{h}, k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right) \\
& \quad=Q^{-1}\left(z, \mathbf{x}, \mathbf{h}, k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right) . \tag{17}
\end{align*}
$$

where $E(z, \mathbf{x}, \mathbf{h})$ is the same extended reflectivity as in equation 7 after coordinate transformation defined by equations 11: $E(z, \mathbf{x}, \mathbf{h})$ $=d(\mathbf{h}) d c h c^{3}(z, \mathbf{x})$.
Equation 18 is to be solved in the direction of increasing two-way time $t$ (thus decreasing $z$ ). Surface single-scattered data (to leading asymptotic order) are obtained by applying an acquisition condition: $d(\mathbf{x}, \mathbf{h}, t)=u(z=0, \mathbf{x}, \mathbf{h}, t)$. Equation 18 generalizes the notion of exploding reflector modeling of zero-offset reflection data (see Lowenthal et al., 1976) to the framework of DSR modeling-imaging (for a detailed discussion, see Appendix B). Here, the extended reflectivity $E(z, \mathbf{x}, \mathbf{h})$ (initial data) can be viewed as a distributed source excited at zero time.
Thus, a possible interpretation of equation 16 is that it describes forward-backward propagation of time slices of $\tilde{u}$, i.e., it allows the recalculation of prestack DSR images as snapshots at different twoway times as schematically illustrated in Figure 4a.

Imaging reflection data, $d=d(\mathbf{x}, \mathbf{h}, t)$, can be formulated as solving the initial value problem for an adjoint to the TWT equation,


Figure 3. Properties of $\mathrm{H}^{\mathrm{TWT}}$ for constant velocity case; (a) slowness surface; (b) group velocity surface.

## Imaging via continuation in two-way time

Note that equation 16 can be interpreted as prestack image continuation with two-way time that is equivalent to applying the timeshifted imaging condition by Sava and Fomel (2006). In this subsection, we reformulate seismic data modeling and imaging, subject to the single-scattering approximation, in terms of solving Cauchy initial value problems in two-way time. In this formulation, modeling takes the form of exploding reflector. A prestack image (extended reflectivity) can be considered a spatially distributed source that is excited simultaneously at zero two-way time. The generated wavefield then propagates along $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays and becomes data whenever these rays reach the acquisition surface.

We now consider the TWT initial value problem for equation 16,

$$
\begin{gather*}
{\left[\partial_{t}-\mathrm{i} P^{\mathrm{TWT}}\left(z, \mathbf{x}, \mathbf{h}, D_{z}, \mathbf{D}_{x}, \mathbf{D}_{h}\right)\right] \widetilde{u}=0,} \\
\left.\widetilde{u}(z, \mathbf{x}, \mathbf{h}, t)\right|_{t=0}=E(z, \mathbf{x}, \mathbf{h}) \tag{18}
\end{gather*}
$$



Figure 4. Forward/backward two-way time continuation for modeling and imaging. (a) Solutions to initial value problems for the homogeneous equation 16 allows recalculating a prestack DSR image from a zero two-way traveltime slice to a given value of two-way traveltime (exploding reflector concept); (b) prestack DSR imaging can be implemented by solving an initial value problem with the inhomogeneous equation 19 (surface data becomes a source function).

$$
\begin{gather*}
(-)\left[\partial_{t}-\mathrm{i} P_{1}^{\mathrm{TWT}}\left(z, \mathbf{x}, \mathbf{h}, D_{z}, \mathbf{D}_{x}, \mathbf{D}_{h}\right)\right] \tilde{u}=d(z) d(\mathbf{x}, \mathbf{h}, t), \\
\left.\widetilde{u}(z, \mathbf{x}, \mathbf{h}, t)\right|_{t=T}=0, \tag{19}
\end{gather*}
$$

where $T$ denotes a sufficiently large two-way time beyond which the data are equal to zero. Equation 19 is to be solved in the direction of decreasing two-way time $t$ (thus increasing $z$ ). The final extended image corresponds to the solution at zero two-way time: $\widetilde{u}(z, \mathbf{x}, \mathbf{h}, t$ $=0$ ). The kinematics (propagation of singularities) of this equation is described by the Hamiltonian defined by equation 12 and the corresponding $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays are defined by equations 14. The approach to imaging using data as a source term for the TWT equation is schematically illustrated in Figure 4b.

## MULTISUBSURFACE-OFFSET ISOCHRON RAYS IN PRESTACK (MAP) MIGRATION

## $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron as an impulse response to a migration operator

In this subsection, we describe Kirchhoff-type implementation of the DSR migration producing prestack DSR images rather than conventional images.

The kinematics of DSR migration is described in Stolk et al. (2009). Here, we provide an interpretation of these kinematics in terms of impulse response surfaces. Fixing a point in $(\mathbf{h}, \mathbf{x}, z)$ space, we can trace DSR rays in all directions. Two-way traveltimes along DSR rays reaching the surface will make up a prestack diffraction surface. All surfaces corresponding to a starting point with $\mathbf{h} \neq 0$ will be nonphysical because they imply source and receiver rays starting from different points in the subsurface at nonzero initial two-way time. These surfaces will not result in coherent summation while using a correct migration velocity. They will come into play only when an incorrect velocity model is used.

One can consider Kirchhoff-type DSR migration which smears multioffset data samples along $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons. For a ray-based construction of an $\mathbf{M}_{s} \mathrm{O}$ isochron, we choose a data sample $d\left(\mathbf{h}_{0}, \mathbf{x}_{0}, t_{0}\right)$ and consider $\widetilde{\mathbf{X}}_{0}=\left(z_{0}=0, \mathbf{h}_{0}, \mathbf{x}_{0}\right)$ as a point source for the Hamilton system of equations 15 . We shoot rays from this point in all directions (defined by $\widetilde{\mathbf{K}}_{0}$ ), starting from initial time $t_{0}$ and solving the system of equations 15 in decreasing two-way time till $t=0$. Taking


Figure 5. Notion of isochrons. (a) $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons for two-way traveltimes $t_{0}=1,3,5$; (b) conventional CO isochrons (thick gray curves) and isochron rays (thick curves) in the image space obtained from $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons and rays by restricting $h$ to zero.
into account that the Hamiltonian defined by equation 13 is independent of $t$ and an endpoint will always correspond to $t=0$, we can then parameterize the distance along $\mathrm{a}_{\mathrm{s}} \mathrm{O}$ isochron ray obtained by solving equations 14 with $t_{0}($ instead of $t)$ :

$$
\begin{align*}
& \widetilde{\mathbf{X}}\left(t_{0}\right)=\left(z\left(0 ; t_{0}, \widetilde{G}_{0}\right), \mathbf{x}\left(0 ; t_{0}, \widetilde{G}_{0}\right), \mathbf{h}\left(0 ; t_{0}, \widetilde{G}_{0}\right)\right), \\
& \widetilde{\mathbf{K}}\left(t_{0}\right)=\left(k_{z}\left(0 ; t_{0}, \widetilde{G}_{0}\right), \mathbf{k}_{x}\left(0 ; t_{0}, \widetilde{G}_{0}\right), \mathbf{k}_{h}\left(0 ; t_{0}, \widetilde{G}_{0}\right)\right) . \tag{20}
\end{align*}
$$

An $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron (an impulse response surface for the prestack depth migration operator) corresponding to two-way time $t_{0}$ follows to be a surface connecting end points of a fan of $\mathrm{M}_{5} \mathrm{O}$ isochron rays $\tilde{\mathbf{X}}\left(t_{0}\right)$ in space $(z, \mathbf{h}, \mathbf{x})$. In Figure 5, we show an example for a 2 D constant velocity model. Thus, midpoint $x$ and offset $h$ are scalars now, while setting $c(z, x)=1$. In Figure 5 a, we show three $\mathrm{M}_{s} \mathrm{O}$ isochrons for the point source $\left(z_{0}, x_{0}, h_{0}\right)=(0,0, .5)$ and two-way time $t_{0}$ being 1,3 and 5, respectively. In Figure 5b, we show some rays (thin curves) from the same point source in the ( $z, x, h$ ) prestack imaging domain.

## Prestack map migration using $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays

Given traveltimes and slopes in the data $d(\mathbf{h}, \mathbf{x}, t)$, we essentially have $\widetilde{\mathrm{G}}_{0}$ — initial data for tracing $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays solving equations 14 . Then we can trace these rays with decreasing time and finally get a prestack image as a union of all end points of the rays. Essentially, we describe map migration as a focusing of $\mathrm{M}_{s} \mathrm{O}$ isochron rays at zero two-way time. The map migration relation in phase space then takes the form
$\left(t_{0}, \widetilde{G}_{0}\right)=\left(t_{0}, z_{0}=0, \mathbf{x}_{0}, \mathbf{h}_{0}, k_{z 0}, \mathbf{k}_{x 0}, \mathbf{k}_{h 0}\right)!\quad\left(t=0, \widetilde{\mathbf{X}}\left(t_{0}\right), \widetilde{\mathbf{K}}\left(t_{0}\right)\right)$,
where $\left(\tilde{\mathbf{X}}\left(t_{0}\right), \widetilde{\mathbf{K}}\left(t_{0}\right)\right)$ are solutions as described in equation 20 .
To get a prestack DSR image, we extract from the output of the relation described by equation 21 only the coordinates $\widetilde{\mathbf{X}}\left(t_{0}\right)$ :

$$
\begin{equation*}
\left(t_{0}, \widetilde{G}_{0}\right)!\quad \widetilde{\mathbf{X}}\left(t_{0}\right)=(z, \mathbf{x}, \mathbf{h}) \tag{22}
\end{equation*}
$$

where $z=z\left(0 ; t_{0}, \widetilde{G}_{0}\right), \mathbf{x}=\mathbf{x}\left(0 ; t_{0}, \widetilde{G}_{0}\right)$, and $\mathbf{h}=\mathbf{h}\left(0 ; t_{0}, \widetilde{G}_{0}\right)$.

## RELATION BETWEEN $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ AND CO ISOCHRONS AND ISOCHRON RAYS

We note that only the section $\mathbf{h}=0$ of the prestack DSR imaging volume $(z, \mathbf{h}, \mathbf{x})$ is important for conventional imaging. Only $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays that have two-way time $t=0$ at $\mathbf{h}=0$ have a direct physical meaning in the framework of wave-scattering theory, i.e., they correspond to a couple of conventional rays starting from the same subsurface point $(\mathbf{x}, z)$ at zero two-way time. One can recognize in equations $t=0$ and $\mathbf{h}$ $=0$ an imaging condition for getting a conventional image. Thus, an intersection of the $\mathrm{M}_{s} \mathrm{O}$ isochron with the $\mathbf{h}=0$ plane is a conventional CO isochron used in common-offset migration as illustrated in Figure 5 (for a 2D constant background velocity case). In Figure 5b, thick gray
curves show CO isochrons corresponding to two lower $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons in Figure 5a $\left(\left(z_{0}, x_{0}, h_{0}\right)=(0,0, .5), t_{0}=3\right.$ and 5$)$. We note that $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons exist for every (positive) $t_{0}$ (as one can see from Figure 5a), while they will not intersect the $\mathbf{h}=0$ plane for $t_{0}$ less than the direct-wave traveltime from source to receiver. Indeed, CO isochrons do not exist for $t_{0}$, 1 in the model example shown in Figure 5 b. We note that this fact causes some complications with initializing CO isochron rays (see Iversen, 2004).

We now establish the connection between $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays and CO isochron rays (in image space) for common-offset migration as defined in Iversen (2004). In particular, we will consider the case of combined parametrization when rays are parameterized by a fixed traveltime slope in a common-offset data section, i.e., $k_{x}$ is kept fixed.

For simplicity, we consider the 2D case so that we can use Figure 5 as an illustration. Every source-receiver pair in an acquisition system can be identified with a point source $\left(x_{0}, h_{0}, z_{0}=0\right)$ for tracing $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays. The direction of an $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron ray is determined by the wave vector $\widetilde{\mathbf{K}}_{0}=\left(k_{x 0}, k_{h 0}, k_{z 0}\right)$. Keeping $k_{x 0}$ fixed according to the combined-parameterizations requirement in Iversen (2004), we can still vary the $k_{h 0}$ component (note that the third component $k_{z 0}$ cannot be chosen arbitrarily but should be found from the equation $\mathrm{H}^{\mathrm{TWT}}=0$ ). Then, we get a oneparameter family of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays as shown in Figure $5 b$ by thin curves. In this case, the family of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays intersects the conventional image plane $h=0$ not more than one time, and the intersection points form a smooth curve, illustrated in Figure 5b by a thick black curve. One can check numerically that this curve coincides with a CO isochron ray in the combined parametrization from Iversen (2004).

It is not at all immediate that the curve obtained by connecting the intersection points of $\mathrm{M}_{5} \mathrm{O}$ isochron rays forms a ray again. We have proven this here in the case of absence of caustics, while constructing a Hamiltonian given in equation A-11 in Appendix A. One can check, however, whether a given set of curves corresponds to a system of rays without constructing a Hamiltonian; one needs to check an integrability condition for the set of curves under investigation, following Duchkov et al. (2008, sec. 5.3).

## $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons in the presence of caustics: A numerical study

We consider a velocity model with a low-velocity lens embedded in a homogeneous medium:

$$
\begin{equation*}
c(x, z)=1-.4 e^{-9\left(x^{2}+(1-z)^{2}\right)} . \tag{23}
\end{equation*}
$$

This model generates caustics in the propagating wavefield but still satisfies the DSR condition: the low-velocity lens is not strong enough to produce turning rays for comparatively short offsets. CO isochrons were constructed for this medium in Stolk (2002, Figure 5). The isochron shown in that figure corresponds to initial parameters $\left(t_{0}, x_{0}, h_{0}, z_{0}\right)=(4.73,0, .2,0)$ and has a rather complicated form. The $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron surface for these initial data is shown in Figure 6a and a few $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays are shown by thin curves in Figure 6b.

Despite the fact that $\mathrm{M}_{s} \mathrm{O}$ isochrons now have complicated forms, the way we construct CO isochrons and isochron rays remains the same. A CO isochron is an intersection of the $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron surface in Figure 6a with a conventional image plane corresponding to $h$ $=0$. This intersection is shown as a gray curve in Figure 6b. A better view of this curve is shown in Figure 7, labeled as $t_{0}=4.72$. Another two CO isochrons are shown with gray curves as well for $t_{0}=2$ and $t_{0}=3.8$. One can see that for small two-way times, CO isochron is a connected curve $\left(t_{0}=2\right)$. Then, a second piece appears around $t_{0}=3.8$, and for two-way time $t_{0}=4.72$ one can see nine smooth branches making up two piece-wise disjoint figures.

For obtaining a CO isochron ray (Iversen, 2004, combined parametrization), we shoot a one-parameter family of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays corresponding to a fixed $k_{x 0}$ as shown in Figure 6 b by thin black curves for $k_{x 0}=.2$. The thick black line corresponds to a CO isochron ray; it is an intersection of a fan of $\mathrm{M}_{s} \mathrm{O}$ isochron rays with an $h$ $=0$ plane. A more complicated case of a CO isochron ray, corresponding to $k_{x 0}=.02$, is shown in Figure 7. We observe that it consists of two disjoint branches (thick black curves) that can appear


Figure 6. $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons and rays in case of caustics. (a) $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron for two-way traveltime $t_{0}=4.72 \mathrm{~s}$; (b) conventional CO isochron (thick gray curve) and isochron ray (thick black curve) in the image space corresponding to slice $h=0$; corresponding $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays are shown by thin curves.


Figure 7. Conventional CO isochrons and an isochron ray in the presence of caustics. Gray curves - CO isochrons as intersection of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron surfaces with $h=0$ plane. They correspond to twoway time $t_{0}=2,3.8$, and 4.7 s (from top to bottom); note that the lowest one is identical to the one in Stolk (2002, Figure 5, bottom). Thick black curves - CO isochron ray as intersection of a fan of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays (for fixed $k_{x 0}=.02$ ) with $h=0$ plane.


Figure 8. Kinematic data for constant background velocity. (a) Dipping reflector; (b) point scatterer; (c) reflected-wave traveltime surface for model (a); (d) scattered-traveltime surface for model (b).


Figure 9. Geometry of the DSR map migration of synthetic data (see Figure 8). (a-c) For the dipping reflector model (Figure 8a); (d-f) for point scatterer (Figure 8b); (a and d) show $\mathrm{M}_{\mathbf{s}} \mathrm{O}$ isochron rays in case of true model ( $c=1$ ), these rays perfectly focus at zero-offset plain; (b and e) show $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays in case of undermigration ( $c$ $=.8$ ); (c and f) end points of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays are connected to form a prestack DSR image. Thick solid curves (points) correspond to map migration with the true model ( $c=1$ ), undermigrated images ( $c=.8$, upper surfaces), and overmigrated images ( $c=1.2$, lower surfaces).
when $\mathrm{M}_{s} \mathrm{O}$ isochron rays intersect the plane $h=0$ more than once. The second branch of the CO isochron ray appears around $t_{0}=3.8$, when the CO isochron splits into two pieces.

We note that caustics generically appear in wave propagation. Thus, the phenomenon of CO isochron-ray branching will generically appear in other heterogenous models.

## EXAMPLES

## Geometry of prestack depth migration as focusing of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays

We illustrate the map migration procedure (described by equations 21 and 22) for a 2D case with constant background velocity, $c(x, z)=1$. We consider two synthetic models here: a dipping plane reflector (Figure 8a) and a point scatterer (Figure 8b). Kinematic data (traveltime surfaces) are shown in Figure 8c and d, respectively. We suppose that traveltimes $\left(x_{0}, h_{0}, t_{0}\right)$ and slopes $\left(k_{x 0}, k_{h 0}\right)$ are picked for these data sets, i.e., that we have a set $\widetilde{G}_{0}$.

We illustrate the DSR map migration in Figure 9 for the dipping reflector model (Figure 9a-c) and the point scatterer model (Figure $9 \mathrm{~d}-\mathrm{f})$. We trace $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays into the subsurface as shown in Figure 9 a and d for the true background velocity. One can see that rays are indeed focusing on a true image in this case. When a wrong background velocity is used to initialize and trace isochron rays, they will defocus as shown in Figure $9 b$ and e. The end points of these rays can be connected to a surface - a prestack migrated image in $(z, x, h)$ coordinates as described by the mapping described by equation 22. In Figure 9c and f, we show prestack DSR images corresponding to undermigration ( $c=.8$, upper surfaces), true velocity model ( $c=1$, thick curve and a point), and overmigration $(c=1.2$, lower surfaces).

## Geometric construction of dip-angle gathers using $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays

Tracing $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays described by equations 14 also provides us with information on the local geometry of the reflector or scatterer through equation 21 . The vector $\widetilde{\mathbf{K}}$ is related to a local dip angle. Following Landa et al. (2008), we can extract the geometrical construction of dip-angle gathers from equation 21 according to

$$
\begin{equation*}
\left(t_{0}, \widetilde{G}_{0}\right)!\quad(z, \mathbf{x}, \boldsymbol{a}), \tag{24}
\end{equation*}
$$

where $z=z\left(0 ; t_{0}, \widetilde{G}_{0}\right), \mathbf{x}=\mathbf{x}\left(0 ; t_{0}, \widetilde{\boldsymbol{G}}_{0}\right)$, and $\boldsymbol{a}=\arctan \left(k_{z}\left(0 ; t_{0}, \widetilde{G}_{0}\right) /\right.$ $\mathbf{k}_{x}\left(0 ; t_{0}, \widetilde{G}_{0}\right)$ ).

We use the same $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays but now to visualize wavenumber information at their end points as defined by the mapping described by equation 24 . After tracing $\mathrm{M}_{s} \mathrm{O}$ isochron rays in the true background velocity ( $c=1$ ), we can visualize dip-angle gathers as shown by solid thick curves in Figure 10. Figure 10a corresponds to a dipping reflector model (see Figure 8a), whereas Figure 10b corresponds to the point scatterer model (see Figure 8b). We also show undermigrated dip-angle gathers ( $c=.8$, upper surfaces) and overmigrated dip-angle gathers ( $c=1.2$, lower surfaces). This geometric representation confirms the observation made in Landa et al. (2008) that a diffracted event is migrated into a surface with a wide support in the dip-angle direction $a=\arctan \left(k_{z} / k_{x}\right)$. A reflection event, however, is mapped into a surface concentrated in the a direction. This is still true when a wrong background velocity is used for map migration.

## DISCUSSION

Iversen (2004) introduced CO isochron rays as curves connecting CO isochrons. He proposed different families of rays, namely, in source, receiver, and combined parameterizations. Isochron rays as curves normal to isochrons were proposed in Silva and Sava (2008) with the purpose to implement common-offset map migration. We note that in the context of map migration, it is necessary to check en-ergy-conservation property along these rays; that is a fundamental property of the ray method (see Kravtsov and Orlov, 1990; Babich and Buldyrev, 1991; Červený, 2001). Only in this case can corresponding isochron rays be used to implement map migration as a propagation of energy from a data space to an image space. A natural way of checking this property is to show that there is a Hamiltonian, such that rays become bicharacteristics, i.e., solutions to a corresponding Hamilton system. In this paper, we have shown that only the CO isochron rays in the combined parameterizations (in the absence of caustics) are solutions to a Hamilton system. In the presence of caustics, it was observed (see Iversen, 2004) that CO isochron rays can branch, with two-way time not changing monotonically. On the contrary, evolution parameters (two-way time in our case) have to grow monotonically along rays which are solutions to a Hamilton system.

We argue that in the framework of DSR migration, the notion of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons and $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays appears naturally in the (geometrical) analysis of this operator. An underlying Hamiltonian (see equation 13) is derived from the DSR equation in a straightforward manner. $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays can be used for implementing DSR map migration after detecting slopes in the data, for example, by slantstack analysis or curvelet/wave-packet decomposition (see Douma and de Hoop, 2007; Chauris and Nguyen, 2008). The initialization of $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays appears as natural as the initialization of rays in geometrical acoustics, unlike the initialization of CO isochron rays as described in Iversen $(2004,2005)$.
$\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons are related to CO isochrons in the same way as DSR migration is related to common-offset migration. A usual motivation for switching from CO to DSR formulation is a necessity to use background velocity models that produce caustics (CO migration produces artifacts in this case).

A DSR imaging volume is of higher dimension as compared to physical space where CO isochrons are defined. In the case of a 2 D physical space, we get a 3D DSR imaging volume with an additional coordinate: subsurface offset. Thus, one has to trace a two-parameter family of $\mathrm{M}_{5} \mathrm{O}$ isochron rays to construct an $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron and a one-


Figure 10. Dip-angle gathers as defined in equation 24. (a) For a dipping reflector model (see Figure 8; (b) for a point scatterer model (see Figure 8). Solid thick curves correspond to map migration with the true model $(c=1)$. One can also see undermigrated gathers ( $c$ $=.8$, upper surfaces) and overmigrated gathers ( $c=1.2$, lower surfaces).
parameter family of CO isochron rays to construct a CO isochron. We note, however, that in the context of map migration, one first gets a sparse data representation while detecting traveltimes and slopes in data. Then, isochron rays are needed only for detected events in data and the computational cost for tracing $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays is not an issue any more. It has furthermore been recognized that it is important to take into account local curvature of isochrons (within a Fresnel zone) while carrying out map-migration style imaging (see Tillmanns and Gebrande, 1999; Luth et al., 2005). One can obtain the isochron curvature from the geometrical spreading associated with $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays.

Furthermore, we give a formulation of DSR migration as a solution to an evolution equation in two-way time (see equations 18 and 19). We can view this formulation as an extension of the exploding reflector concept from ZO to DSR migration using a single evolution equation. (An alternative extension, using two wave operators, was considered by Biondi (2006b)).

## CONCLUSIONS

In this paper, we show that isochrons defined by imaging operators can be identified with wavefronts of solutions to an evolution equation. Rays associated with this equation provide a natural way of implementing prestack map migration. In the absence of caustics, properly chosen CO isochron rays can be used for CO map migration. Explicit formulas for these rays (and corresponding Hamiltonians) are given in the case of constant velocity models or in the framework of time migration. In the presence of caustics, we succeeded in formulating DSR map migration in terms of a flow-out in two-way time. $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons and $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays appear naturally as solutions to the corresponding Hamiltonian.
$\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays are defined in the DSR imaging volume with the coordinates subsurface midpoint, offset, two-way time, and depth, and are generated by a Hamiltonian. Our construction of this Hamiltonian depends on the DSR condition, which requires rays to be nowhere horizontal. Two-way time is used as an evolution parameter along $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays. Then, the end points of these $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochron rays at zero two-way time form a DSR prestack image.

We presented an evolution equation in two-way time that can be used for DSR imaging as an alternative to the DSR equation for downward data continuation in depth. $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons appear as wavefronts associated with solutions to an evolution equation. We emphasize that both equations are kinematically equivalent (have the same rays). However, they are two different representations of the same imaging operator while using different evolution parameters along rays. This results in differences in implementation; for example, an imaging condition is not required when data are propagated in two-way time.
Finally, we give some examples illustrating the geometry of a prestack DSR migration for two synthetic models: a plane dipping reflector and a point scatterer (both in a constant velocity background medium). We establish how CO isochrons can be related to $\mathrm{M}_{\mathrm{s}} \mathrm{O}$ isochrons through a restriction to zero subsurface offset.

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## APPENDIX A

## HAMILTONIAN FOR COMMON-OFFSET ISOCHRON RAYS

In this appendix, we derive a Hamiltonian for common-offset (CO) isochron rays in the absence of caustics following the derivation in Duchkov et al. (2008, sec. 5.1). We consider the 2D case, when midpoint coordinate $x_{m}$ and horizontal coordinates $x$ and $x_{0}$ are scalars, and surface offset $h$ is a fixed parameter. We express migra-tion-demigration in terms of Kirchhoff-type integral operators (see Beylkin, 1985; Bleistein, 1987).

The integral kernel of a demigration operator $F$, which maps an image $w\left(x_{0}, z_{0}\right)$ to CO data $u\left(x_{m}, t^{\prime}\right)$, takes the standard form:

$$
\begin{equation*}
K_{F}\left(x_{m}, t^{\prime} ; x_{0}, z_{0}\right)=\int a\left(x_{m}, x_{0}, z_{0}\right) e^{\operatorname{iv}\left(T\left(x_{m}, x_{0}, z_{0}\right)-t^{\prime}\right)} \mathrm{d} v^{\prime} \tag{A-1}
\end{equation*}
$$

where $T\left(x_{m}, x_{0}, z_{0}\right)$ is the two-way time along rays connecting scattering point $\left(x_{0}, z_{0}\right)$ to a source at $x_{m}-h$ and a receiver at $x_{m}+h$.

We introduce a shifted demigration operator $F\left(a_{0}\right)$ that maps an image $w\left(x_{0}, z_{0}\right)$ to CO data $u\left(x_{m}, t^{\prime}\right)$ shifted in the time direction by $a_{0}$. Its adjoint is a shifted migration operator $F^{*}(a)$ that first shifts data $u\left(x_{m}, t^{\prime}\right)$ in time by $\boldsymbol{a}$ and then migrates the result to an image $w(x, z)$. The corresponding operator kernels take the form:

$$
\begin{equation*}
K_{F}(a)\left(x_{m}, t^{\prime} ; x_{0}, z_{0}\right)=\int a\left(x_{m}, x_{0}, z_{0}\right) e^{\mathrm{i} \boldsymbol{v}^{\prime}\left(T\left(x_{m}, x_{0}, z_{0}\right)-\left(t^{\prime}-a\right)\right)} \mathrm{d} \boldsymbol{v}^{\prime} \tag{A-2}
\end{equation*}
$$

$$
\begin{equation*}
K_{F^{*}}(a)\left(x, z ; x_{m}, t^{\prime}\right)=\int a \overline{\left(x_{m}, x, z\right)} e^{-\mathrm{i} v\left(T\left(x_{m}, x, z\right)-\left(t^{\prime}-a\right)\right)} \mathrm{d} v . \tag{A-3}
\end{equation*}
$$

In the absence of caustics, we can compose the shifted migration operator with the shifted demigration operator. The kernel for the composition, $F^{*}(a) F\left(a_{0}\right)$, after integrating out $v^{\prime}$ and $t^{\prime}$, follows to be

$$
\begin{align*}
& K_{F * F}\left(a_{0}, a\right)\left(x, z ; x_{0}, z_{0}\right) \\
& \quad=\int b\left(x, z, x_{m}, x_{0}, z_{0}\right) e^{\mathrm{i} W\left(a_{0}, a, v, x_{m}, x, z, x_{0}, z_{0}\right)} \mathrm{d} v \mathrm{~d} x_{m}, \tag{A-4}
\end{align*}
$$

where the phase function is given by

$$
\begin{align*}
& w\left(a_{0}, a, v, x_{m}, x, z, x_{0}, z_{0}\right) \\
& \quad=\quad v\left[T\left(x_{m}, x_{0}, z_{0}\right)-T\left(x_{m}, x, z\right)+a_{0}-a\right] . \tag{A-5}
\end{align*}
$$

We do not further specify the amplitude function $b$ because we consider the kinematics only, here.

The set of stationary points of $w$, to be used in a stationary phase approximation of the right-hand side of equation A-5, is defined by the equations,

$$
\partial_{v} w=T\left(x_{m}, x_{0}, z_{0}\right)-T\left(x_{m}, x, z\right)+a_{0}-a=0
$$

$$
\begin{equation*}
v^{-1} \partial_{x_{m}} w=\partial_{x_{m}} T\left(x_{m}, x_{0}, z_{0}\right)-\partial_{x_{m}} T\left(x_{m}, x, z\right)=0 \tag{A-6}
\end{equation*}
$$

and leads to the construction of the relevant bicharacteristics and common-offset isochron rays. To obtain a Hamiltonian for these rays, we follow the derivation in Duchkov et al. (2008, sec. 5.1). In particular, we introduce the generating functions $S^{\prime}\left(a ; x, z, x_{m}, v\right)$ $=v\left(T\left(x_{m}, x, z\right)+a\right) \quad$ and $\quad \widetilde{S}\left(a ; x, z, k_{x_{m}}, v\right)=v\left(T\left(x_{m}, x, z\right)+\boldsymbol{a}\right)$ $+\left\langle k_{x_{m}}, x_{m}\right\rangle$ as described there. Equation 54 in Duchkov et al. (2008) takes the form

$$
\begin{equation*}
v \partial_{x} T\left(x_{m}, x, z\right)=k_{x}, \quad v \partial_{z} T\left(x_{m}, x, z\right)=k_{z} \tag{A-7}
\end{equation*}
$$

and can be solved for $x_{m}\left(x, z, k_{x}, k_{z}\right)$ and $v\left(x, z, k_{x}, k_{z}\right)$ in the case of absence of caustics. We note that $\partial_{a} \widetilde{S}\left(a ; x, z, k_{x_{m}}, v\right)=v$, and obtain the Hamiltonian (Duchkov et al., 2008, equation 55)

$$
\begin{equation*}
\mathrm{H}^{\mathrm{CO}}\left(x, z, k_{a}, k_{x}, k_{z}\right)=k_{a}-v\left(x, z, k_{x}, k_{z}\right) . \tag{A-8}
\end{equation*}
$$

This Hamiltonian can be used for tracing bicharacteristics in heterogeneous media in the absence of caustics.

We honor the fact that the evolution parameter $a$ corresponds to a shift in two-way traveltime and identify $a \equiv t, k_{a} \equiv v$. Then, the Hamiltonian in equation A-8 describes continuation in time and generates the Hamilton flow transforming an isochron corresponding to two-way traveltime $t_{0}$ into another isochron corresponding to twoway traveltime $t$, while midpoint and offset are not changing (this can be verified by analyzing the composition of operators). Thus, the Hamiltonian $\mathrm{H}^{\mathrm{CO}}$ describes an evolution of isochrons as discussed in Iversen (2004, equation 23). The rays corresponding to the Hamiltonian defined by equation A-8 coincide with those from Iversen (2004) in the combined parametrization.

The Hamiltonian does not depend on $t$, and thus $v$ is preserved by the Hamilton flow. Equation A-7 results in the slowness vector ( $k_{x} / v, k_{z} / v$ ) being normal to an isochron (and coincides with the introduction of slowness vector in Iversen (2004, equation 23). Equation A-6 expresses that the traveltime slope $\partial_{x_{m}} T\left(x_{m}, x, z\right)=k_{x_{m}} / V$ is preserved along the ray (see Iversen, 2004, equation 34).

Then we denote CO isochron rays by

$$
\begin{equation*}
\left(x\left(t ; t_{0}, g_{0}\right), z\left(t ; t_{0}, g_{0}\right), k_{x}\left(t ; t_{0}, g_{0}\right), k_{z}\left(t ; t_{0}, g_{0}\right)\right), \tag{A-9}
\end{equation*}
$$

which solve the Hamilton system

$$
\begin{equation*}
\frac{\mathrm{d}(x, z)}{\mathrm{d} t}=\frac{\partial \mathrm{H}^{\mathrm{CO}}}{\partial\left(k_{x}, k_{z} z\right.}, \quad \frac{\mathrm{d}\left(k_{x}, k_{z}\right)}{\mathrm{d} t}=-\frac{\partial \mathrm{H}^{\mathrm{CO}}}{\partial(x, z)} \tag{A-10}
\end{equation*}
$$

for initial conditions $g_{0}=\left(x_{0}, z_{0}, k_{x 0}, k_{z 0}\right)$, the starting point and orientation of the ray. Here, two-way time $t$ is the evolution parameter.

Also, one can solve the associated dynamic ray-tracing system along these rays providing a geometrical spreading. It was shown in Iversen (2004) that this geometrical spreading is related to the Beylkin determinant, a correction factor in amplitude-preserving imaging. It also provides a local curvature of an isochron (front) that can be used while implementing a map-migration style imaging as proposed in Tillmanns and Gebrande (1999) and Luth et al. (2005).

In the case of a constant velocity v , we get an explicit formula for the Hamiltonian:

$$
\begin{equation*}
\mathrm{H}^{\mathrm{CO}}\left(x, z, v, k_{x}, k_{z}\right)=v-\frac{\mathrm{v}}{k_{x} z \sqrt{2}}\left(\frac{\sqrt{Q_{-} Q_{+}}}{\sqrt{Q_{-}}+\sqrt{Q_{+}}}\right) \tag{A-11}
\end{equation*}
$$

in which

$$
\begin{align*}
Q_{ \pm} & =z^{2}\left(k_{x}^{2}+k_{z}^{2}\right)^{2}+\left(2 h k_{x} k_{z} \pm z\left(k_{x}^{2}-k_{z}^{2}\right)\right) q_{ \pm} \\
q_{ \pm} & =2 h k_{x} k_{z} \pm \sqrt{4 h^{2} k_{x}^{2} k_{z}^{2}+z^{2}\left(k_{x}^{2}+k_{z}^{2}\right)^{2}} \tag{A-12}
\end{align*}
$$

In the zero-offset case ( $h=0$ ), one can check that $\mathrm{H}^{\mathrm{zO}}\left(x, z, v, k_{x}, k_{z}\right)$ $=v-\mathrm{v} / 2 \sqrt{k_{x}^{2}+k_{z}^{2}}$, and we recover the exploding reflector concept as further discussed in Appendix B (see equation B-3).

## APPENDIX B

## EXPLODING REFLECTOR MODEL

For vertically inhomogeneous velocity models, we have $c\left(z, \mathbf{x}_{r}\right)$ $=c\left(z, \mathbf{x}_{s}\right)=c(z)$. In this case, $c_{-}^{2}=0$ everywhere and equation 10 needs to be modified to avoid the appearance of a singularity. For this specific case, it is straightforward to rederive the Hamiltonian by solving equation 9 again:

$$
\begin{align*}
& \mathrm{H}^{\mathrm{TWT}}\left(z, \mathbf{x}_{s}, \mathbf{x}_{r}, v, k_{z}, \mathbf{k}_{s}, \mathbf{k}_{r}\right) \\
& \quad= \pm\left(v-k_{z} \frac{c(z)}{2} \sqrt{k_{z}^{-4}\left(\left\|\mathbf{k}_{s}\right\|^{2}-\left\|\mathbf{k}_{r}\right\|^{2}\right)^{2}+2 k_{z}^{-2}\left(\left\|\mathbf{k}_{r}\right\|^{2}+\left\|\mathbf{k}_{s}\right\|^{2}\right)+1}\right) . \tag{B-1}
\end{align*}
$$

Transforming the Hamiltonian from subsurface lateral source and receiver coordinates to subsurface midpoint and offset coordinates yields

$$
\begin{align*}
& \mathrm{H}^{\mathrm{TWT}}\left(z, \mathbf{x}, \mathbf{h}, v, k_{z}, \mathbf{k}_{x}, \mathbf{k}_{h}\right) \\
& \quad=v-k_{z} \frac{c(z)}{2} \sqrt{k_{z}^{-4}\left\langle\mathbf{k}_{x}, \mathbf{k}_{h}\right\rangle^{2}+k_{z}^{-2}\left(\left\|\mathbf{k}_{x}\right\|^{2}+\left\|\mathbf{k}_{h}\right\|^{2}\right)+1} \tag{B-2}
\end{align*}
$$

We note that for constant background velocity, for example $c(z)$ $=\mathrm{v}$, our equation B-1 reduces to Sava's (2003) equation 3; our equation B-2 reduces to the Fourier domain counterpart of Fomel's (2003b) equation A-10.

We use the Hamiltonians for data continuation in two-way traveltime and depth to revisit the exploding reflector model used in the early development of seismic imaging. For a vertically inhomogeneous velocity model, $c(z)$, the Hamiltonian $\mathrm{H}^{\text {TWT }}$ in equation $\mathrm{B}-2$ does not depend on $\mathbf{h}$, and thus the phase variable $\mathbf{k}_{h}$ remains constant in the course of downward continuation $\left(\mathrm{d} \mathbf{k}_{h} / \mathrm{d} t=\right.$ $-\partial H^{\text {TWT }} / \partial \mathbf{h}=0$ ). For zero source-receiver offset (ZO) surface data, $\mathbf{k}_{h}=0$ (this follows immediately from the symmetry of common midpoint gathers) and it remains zero for all $t$. Then, equation B-2 reduces to the Hamiltonian for zero-offset data modeling in the exploding-reflector approach by Lowenthal et al. (1976), Claerbout (1985), and Cheng and Coen, (1984):

$$
\begin{equation*}
v-\frac{c(z)}{2} \sqrt{k_{z}^{2}+\left\|\mathbf{k}_{x}\right\|^{2}}=\mathrm{H}^{\mathrm{zO}}\left(\mathbf{x}, z, v, \mathbf{k}_{x}, k_{z}\right) \tag{B-3}
\end{equation*}
$$

where we recognize the half-velocity $\frac{1}{2} c(z)$ typical for the exploding reflector model. We note that we describe, here, the exploding reflector model by a first-order evolution equation instead of a second-order wave equation.

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