Chapter 2

The method of stationary phase

In this chapter we introduce a very useful analytical tool which will allow us to find asymptotic expansions for integrals that cannot, in many cases, be calculated in any other way. As the chapter title indicates, this tool is known as the "method of stationary phase". The general objective of the method is to find an asymptotic expression as $\lambda \to \infty$ for integrals of the form

$$I_{a_\lambda,\phi}(y,\lambda) = \int e^{i\lambda\phi(x,y)} a_\lambda(x,y)\,\mathrm{d}x. \tag{2.1}$$

This means that our goal is to find expressions of the form

$$|I_{a_\lambda,\phi}(y,\lambda) - I(y,\lambda)| = O(\lambda^{-N})$$

(or some variation on this) where $I(y,\lambda)$ is calculated explicitly. The intuition here is that when λ becomes very large $\lambda\phi(x,y)$ oscillates rapidly with respect to x and these oscillations will cancel out in the integral except at stationary points of ϕ (i.e. points where $\partial_x\phi = 0$). Thus the integral only depends on a_λ and its derivatives at these stationary points. We will now proceed to make this rigorous.

2.1 Assumptions and non-stationary case

We begin by introducing assumptions about ϕ and a_λ. Let $U \subset \mathbb{R}^{\tilde{n}}$ be open. We will build up results that gradually require stronger and stronger hypotheses on ϕ, but to begin we only suppose that $\phi \in C^\infty(\mathbb{R}^n_x \times U_y)$. Also, we assume that for every fixed λ, $a_\lambda \in C^\infty(\mathbb{R}^n_x \times U_y)$. Of course we must also have a hypothesis controlling the growth of a_λ as $\lambda \to \infty$. To do this we introduce the following definition.

Definition 2.1 (Hypotheses on a_λ.). *If there exists an $m \in \mathbb{R}$ and $\delta < 1$ such that given any compact set $K \Subset U$ and multi-index α*

$$\sup_{y\in K} \|\partial_x^\alpha a_\lambda(\cdot_x, y)\|_{L^p(\mathbb{R}^n_x)} = O(\lambda^{m+\delta|\alpha|})$$

we will say in this case that $a_\lambda \in I_\delta^{m,p}(\mathbb{R}^n_x \times U_y)$*. We will usually only consider the cases* $p = 1$ *or* ∞.

The first result of the chapter partially confirms the intuition presented in the introduction.

Lemma 2.2. *Suppose that* $\partial_x \phi$ *is not zero anywhere on the support of* $a_\lambda \in I_\delta^{m,1}(\mathbb{R}^n_n \times U_y)$ *for any* λ*. Then for every* $N \in \mathbb{N}$ *and compact set* $K \Subset U$

$$\sup_{y \in K} |I_{a_\lambda,\phi}(y,\lambda)| = O(\lambda^{-N}) \quad \text{as } \lambda \to \infty.$$

Proof. Given the hypotheses we have for any $M \in \mathbb{N}$

$$I_{a_\lambda,\phi}(y,\lambda) = \int_{\mathbb{R}^n} \frac{\langle \partial_x \phi(x,y), \partial_x \rangle^M e^{i\lambda\phi(x,y)}}{(i\lambda |\partial_x \phi(x,y)|^2)^M} a_\lambda(x,y)\,\mathrm{d}x.$$

Now we integrate by parts to obtain

$$\begin{aligned}
\sup_{y \in K} |I_{a,\phi}(y,\lambda)| &\le \sup_{y\in K} \frac{1}{\lambda^M} \int_{\mathbb{R}^n} \left| \left\langle \partial_x, \frac{\partial_x \phi(x,y)}{|\partial_x \phi(x,y)|^2} \right\rangle^M a_\lambda(x,y) \right| \mathrm{d}x \\
&\le C_{M,K} \lambda^{-M} \sum_{|\alpha| \le M} \sup_{y\in K} \|\partial_x^\alpha a_\lambda(\cdot_x, y)\|_{L^1(\mathbb{R}^n_x)} \\
&= O(\lambda^{m+(\delta-1)M}).
\end{aligned}$$

Therefore, by taking $M \ge (N+m)/(1-\delta)$ the proof is complete. □

This lemma settles the matter when there are no stationary points, and so we are now left with the more interesting and sophisticated case of evaluating (2.1) in the case that ϕ has stationary points on the support of a_λ. To begin this we add the additional hypothesis that all stationary points of ϕ are non-degenerate:

Definition 2.3. *If* $\partial_x \phi(x,y) = 0$ *then* x *is called a non-degenerate stationary point of* $\phi(\cdot_x, y)$ *if the Hessian* $\partial_x^2 \phi(x,y)$ *is an invertible matrix.*

In particular this means that for fixed y the set of stationary points of $\phi(\cdot_x, y)$ is discrete, and so by introducing a partition of unity we may reduce the asymptotic evaluation of (2.1) to the case in which ϕ has exactly one non-degenerate stationary point on the support of a_λ, and we may furthermore assume that for all λ and y, $a_\lambda(\cdot_x, y)$ has support contained within an arbitrarily small neighborhood of this stationary point. The idea now is to introduce a local change of coordinates using the so-called Morse Lemma that changes ϕ to a quadratic form. Then we will show how to asymptotically evaluate an integral in the form (2.1) in the case that ϕ is a quadratic form using lemma 1.6.

2.2 Morse Lemma

As mentioned above, we intend to introduce a local change of coordinates in x, smoothly depending on y, near a non-degenerate stationary point that transforms $\phi(x,y)$ into a quadratic form with respect to x. That this is possible is a classical result known as the Morse Lemma, although the classical result generally does not include the parameter y. This proof largely follows one which can be found in [**?**].

Lemma 2.4 (Morse Lemma). *Suppose $y_0 \in U$ and x_0 is a non-degenerate stationary point for $\phi(\cdot_x, y_0)$. Then there exists a neighborhood $V \subset U$ of y_0, a neighborhood W of x_0, a smooth function $X : V \to W$, and a function $\Psi : W \times V \to \mathbb{R}^n$ such that*

1. *For every $y \in V$, $X(y)$ is the unique stationary point, which is also nondegenerate, for $\phi(\cdot_x, y)$ in W.*
2. *For every $y \in V$, the map $W \ni x \mapsto \Psi(x, y)$ is a diffeomorphism onto its image and*

$$\phi(x, y) = \phi(X(y), y) + \frac{1}{2}\Psi(x, y)^T \partial_x^2 \phi(X(y), y)\Psi(x, y). \tag{2.2}$$

Furthermore, $\Psi(X(y), y)) = 0$ and $\partial_x \Psi(X(y), y) = \mathrm{Id}$.

Proof. The existence of the function X with the claimed properties follows from the fact that for given y the stationary points are solutions of the equation

$$0 = \partial_x \phi(x, y).$$

Since x_0 is a non-degenerate stationary point the implicit function theorem shows that this equation implicitly defines x as a function of y in some neighborhood of (x_0, y_0).

To show that the function Ψ with the claimed properties exists we take the Taylor expansion of ϕ in x about $X(y)$ which gives (note that the first derivative term vanishes because we are expanding at a stationary point)

$$\phi(x, y) = \phi(X(y), y) + \frac{1}{2}(x - X(y))^T B(x, y)(x - X(y))$$

where

$$B(x, y) = \int_0^1 (1 - s)\, \partial_x^2 \phi(sx + (1 - s)X(y), y)\, \mathrm{d}s.$$

We now wish to find a smooth matrix valued function $R(x, y)$ such that $\Psi(x, y) = R(x, y)(x - X(y))$ has the desired properties. From the previous formula this will satisfy (2.2) if

$$R^T \partial_x^2 \phi(X(y), y)R - B(x, y) = 0. \tag{2.3}$$

Let us interpret the left hand side of this equation as a mapping from $M_n(\mathbb{R})_R \times \mathbb{R}^n_x \times V_y$ ($M_n(\mathbb{R})$ is the set of $n \times n$ matrices) to the symmetric matrices $\mathcal{S}_n(\mathbb{R})$. If we take the differential of the left hand side with respect to R evaluated at the identity we obtain

$$dR \mapsto (dR)^T \partial_x^2 \phi(X(y), y) + \partial_x^2 \phi(X(y), y)dR$$

which is surjective since when C is symmetric

$$(\frac{1}{2}\partial_x^2 \phi(X(y), y)^{-1}C)^T \partial_x^2 \phi(X(y), y) + \partial_x^2 \phi(X(y), y)(\frac{1}{2}\partial_x^2 \phi(X(y), y)^{-1}C) = C.$$

Therefore, by the implicit function theorem again there exists a smooth matrix valued function $R(x,y)$ defined on some neighborhood of (x_0,y_0) that satisfies (2.3) everywhere that it's defined. Possibly shrinking the neighborhood where this is defined completes the proof except for the derivative of $\partial_x\Psi$. However this is straightforward since

$$\partial_x|_{x=X(y)}\Psi(X(y),y) = (\partial_x R(x,y)(x-X(y)) + R(x,y)|_{x=X(y)} = \mathrm{Id}.$$

□

2.3 Asymptotic formula

Now let us apply the Morse lemma in order to complete the asymptotic evaluation of (2.1). Indeed, we have the following result.

Theorem 2.5. *Suppose that $a_\lambda \in I_\delta^{m,\infty}(\mathbb{R}^n_x\times U_y)\cap I_\delta^{m,1}(\mathbb{R}^n_x\times U_y)$ with $\delta<1/2$ and that $y_0\in U$ and $\phi(\cdot_x,y_0)$ has exactly one non-degenerate stationary point on the support of $a_\lambda(\cdot_x,y_0)$ at x_0. Then there exists a neighborhood $V\subset U$ of y_0 such that*

$$\sup_{y\in V}\left|I_{a_\lambda,\phi}(y,\lambda) - \left(\frac{2\pi}{\lambda}\right)^{n/2}\frac{e^{\mathrm{i}(\mathrm{sgn}(\partial_x^2\phi(X(y),y))\frac{\pi}{4}+\lambda\phi(X(y),y))}}{\sqrt{|\partial_x^2\phi(X(y),y)|}}a_\lambda(X(y),y)\right| = O(\lambda^{m-n/2-(1-2\delta)}).$$

Proof. Take V, W, X, and Ψ to be the sets and mappings guaranteed to exist by the Morse lemma. Then using lemma 2.2 and a partition of unity we can assume without loss of generality that $a_\lambda(\cdot_x,y)$ is supported in W for all $y\in V$. Thus, we can change variables in the integral $I_{a_\lambda,\phi}$ to $\tilde{x}=\Psi(x,y)$ to obtain

$$\begin{aligned} I_{a_\lambda,\phi} &= \int e^{i\lambda\left(\phi(x_0,y_0)+\tilde{x}^T\frac{\partial_x^2\phi(X(y),y)}{2}x\right)} a_\lambda(\Psi^{-1}(\tilde{x},y),y)\,\frac{1}{|\partial_x\Psi(\Psi^{-1}(\tilde{x},y),y)|}\,\mathrm{d}\tilde{x} \\ &= e^{i\lambda\phi(x_0,y_0)}\int e^{i\lambda\tilde{x}^T\frac{\partial_x^2\phi(X(y),y)}{2}x} f_\lambda(\tilde{x},y)\,\mathrm{d}x. \end{aligned}$$

Now, applying $\mathcal{F}^{-1}$ and $\mathcal{F}$ and using the definition of $\mathcal{F}^{-1}$ on $\mathcal{S}'(\mathbb{R}^n)$ we obtain

$$I_{a_\lambda,\phi} = e^{i\lambda\phi(x_0,y_0)}\int \mathcal{F}_{\tilde{x}}\left(e^{i\lambda\tilde{x}^T\frac{\partial_x^2\phi(X(y),y)}{2}x}\right)\mathcal{F}^{-1}_{\tilde{x}}\left(f_\lambda(\tilde{x},y)\right)(\xi)\,\mathrm{d}\xi.$$

Using lemma 1.6 and writing $B(y)=\partial_x^2\phi(X(y),y)$ gives

$$I_{a_\lambda,\phi} = e^{i\lambda\phi(x_0,y_0)}\left(\frac{2\pi}{\lambda}\right)^{n/2}\frac{e^{\mathrm{sgn}(B(y))\frac{\pi i}{4}}}{\sqrt{|B(y)|}}\int e^{-\frac{i}{2\lambda}\xi^t B(y)^{-1}\xi}\mathcal{F}^{-1}_{\tilde{x}}\left(f_\lambda(\tilde{x},y)\right)(\xi)\,\mathrm{d}\xi.$$

Next we use Taylor's theorem to assert that for any integer $M>0$,

$$e^b = \sum_{k=0}^M b^k/k! + b^{M+1}g_{M+1}(b)$$

where $g_{M+1}(b) \in C^\infty(\mathbb{C})$ satisfies $|g_{M+1}(b)| \leq |e^b|/(M+1)!$ and so (1.2) implies that, setting $b = -\frac{i}{2\lambda}\xi^t B(y)^{-1}\xi$,

$$I_{a_\lambda,\phi} - \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{e^{\mathrm{i}(\operatorname{sgn}(B)\frac{\pi}{4}+\lambda\phi(X(y),y))}}{\sqrt{|B|}} \int \sum_{k=0}^{M} \frac{i^k}{(2\lambda)^k k!} \mathcal{F}_{\tilde{x}}^{-1}\left(\langle \partial_{\tilde{x}}, B^{-1}\partial_{\tilde{x}}\rangle^k f_\lambda\right)(\xi)\,\mathrm{d}\xi$$
$$= \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{e^{\mathrm{i}(\operatorname{sgn}(B)\frac{\pi}{4}+\lambda\phi(X(y),y))}}{\sqrt{|B|}} \int \mathcal{F}_{\tilde{x}}^{-1}\left(\left(\frac{\langle \mathrm{i}\partial_{\tilde{x}}, B^{-1}\partial_{\tilde{x}}\rangle}{2\lambda}\right)^{M+1} f_\lambda\right)(\xi)\, g_{M+1}(b)\,\mathrm{d}\xi.$$

Finally, since $\int \mathcal{F}^{-1}(f)(\xi)\,\mathrm{d}x = f(0)$ by the Fourier inversion theorem the above equality implies, using also part 2 of lemma 2.2,

(2.4)
$$\left| I_{a_\lambda,\phi} - \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{e^{\mathrm{i}(\operatorname{sgn}(B)\frac{\pi}{4}+\lambda\phi(X(y),y))}}{\sqrt{|B|}} a_\lambda(X(y),y) \right|$$
$$\leq C\lambda^{-n/2} \sum_{|\alpha|\leq 2M} \lambda^{-\max(\operatorname{ceil}(|\alpha|/2),1)} \left|[\partial_x^\alpha a_\lambda](X(y),y)\right|$$
$$+ C\lambda^{-n/2-(M+1)} \left\| \mathcal{F}_{\tilde{x}}^{-1}\left(\left(\langle \partial_{\tilde{x}}, B^{-1}\partial_{\tilde{x}}\rangle\right)^{M+1} f_\lambda\right)(\xi) \right\|_{L_1(\mathbb{R}^n_\xi)}$$
$$\leq C\lambda^{m-n/2-(1-2\delta)} + C\lambda^{-n/2-(M+1)} \sum_{|\alpha|\leq n+1+2(M+1)} \|\partial_x^\alpha a_\lambda(x,y)\|_{L^1(\mathbb{R}^n_x)}$$
$$\leq C\lambda^{m-n/2-(1-2\delta)} + C\lambda^{m-n/2+(M+1)(2\delta-1)+(n+1)\delta}.$$

The constants C may change from step to step, and in the second inequality we used lemma 1.5. The notation $\operatorname{ceil}(|\alpha|/2)$ just means that we round $|\alpha|/2$ up to the nearest integer. Now, since $\delta < 1/2$ we can take M sufficiently large so that $(M+1)(2\delta-1)+(n+1)\delta < -(1-2\delta)$ and this proves the result.

□

Using the same method as in the proof of theorem 2.5 we could find an asymptotic expression for $I_{a_\lambda,\phi}$ to higher order. However the terms in the expansion become very complicated quite quickly. Nonetheless we will in some cases require explicit expressions for more than the first term of the asymptotic expansion. To get these expressions it will generally be easier to manipulate a_λ (usually using integration by parts), and then apply the next result which is really a corollary of the proof of theorem 2.5.

Corollary 2.6. *Suppose we have the same hypotheses as in theorem 2.5 and use the same notation as in the proof. Further, assume that*

$$\partial_x^\alpha a_\lambda(X(y),y) = 0$$

for all $|\alpha| \leq 2N-1$ *and*

$$[\langle \partial_x, \partial_x^2\phi(X(y),y)^{-1}\partial_x\rangle^N a_\lambda](X(y),y) = 0.$$

Then

$$\sup_{y\in V}|I_{a_\lambda,\phi}(y,\lambda)| = O(\lambda^{m-n/2-(N+1)(1-2\delta)}).$$

Proof. We simply follow the proof of theorem 2.5 until (2.4) which in this case, due to the vanishing of a_λ at the stationary point, can be replaced by

$$|I_{a_\lambda,\phi}(y,\lambda)| \leq C\lambda^{m-n/2-(N+1)(1-2\delta)} + C\lambda^{m-n/2+(M+1)(2\delta-1)+(n+1)\delta}.$$

Once again we take M sufficiently large to establish the result. □