Chapter 1

Oscillatory integrals

1.1 Fourier transform on $\mathcal{S}$

The Fourier transform is a fundamental tool in microlocal analysis and its application to the theory of PDEs and inverse problems. In this first section we review the basic properties of the Fourier transform acting on the Schwartz space. The space of Schwartz functions on $\mathbb{R}^n$, for which we use the notation $\mathcal{S}(\mathbb{R}^n)$, is defined to be those $u \in C^\infty(\mathbb{R}^n)$ for which

$$\|u\|_{\beta,\alpha} = \sup_{x\in\mathbb{R}^n} |x^\beta \partial_x^\alpha u(x)|$$

is finite for all β and $\alpha \in \mathbb{N}^n$. In fact for every α, $\|\cdot\|_{\beta,\alpha}$ provides a semi-norm on $\mathcal{S}$ and once equipped with this collection of semi-norms $\mathcal{S}$ becomes a Frechet space (see for example Folland's Real Analysis).

For $u \in \mathcal{S}(\mathbb{R}^n)$ the Fourier Transform of u is defined as[1]

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x.$$

Differentiating under the integral sign we easily see that

$$\partial_{\xi^j}(\mathcal{F}(u))(\xi) = \mathcal{F}(-ix^j u)(\xi). \tag{1.1}$$

Similarly, we may integrate by parts to show

$$\mathcal{F}(\partial_x^j u)(\xi) = i\xi^j \mathcal{F}(u)(\xi). \tag{1.2}$$

In view of (1.1) and (1.2) we adopt the usual notation $D_{x^j} = -i\partial_{x^j}$ for convenience, and we will sometimes write

$$\hat{u} = \mathcal{F}(u).$$

[1] Readers should be warned that there is no general agreement on precisely how the Fourier transform should defined. For some authors there is no negative sign in front of the i, and factors of 2π sometimes appear in various places.

Using (1.1) and (1.2) we can establish the following.

Lemma 1.1. $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *continuously.*

Proof. For $u \in \mathcal{S}(\mathbb{R}^n)$ we have for any multi-indices

$$\begin{aligned}\|\hat{u}\|_{\beta,\alpha} &= \sup_\xi |\xi^\beta D_\xi^\alpha \hat{u}(\xi)| \\ &= \sup_\xi |\mathcal{F}[D_x^\beta((-x)^\alpha u)](\xi)| \\ &= \sup_\xi \left| \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi}(1+|x|)^{n+1}[D_x^\beta(-x)^\alpha u](x) \frac{\mathrm{d}x}{(1+|x|)^{n+1}} \right| \\ &\leq \left(\sup_x |(1+|x|)^{n+1}[D_x^\beta(-x)^\alpha u](x)| \right) \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{(1+|x|)^{n+1}} \\ &\leq C \sum_{|\beta'| \leq |\alpha+n+1|,\, \alpha' \leq \beta} \|u\|_{\beta',\alpha'}.\end{aligned}$$

with the constant C only depending on the dimension n. This completes the proof. ☐

We will now calculate the Fourier Transform of a Gaussian.

Theorem 1.2.

$$\mathcal{F}(e^{-t|x|^2})(\xi) = \pi^{n/2} \frac{e^{\frac{-|\xi|^2}{4t}}}{t^{n/2}} \quad \forall\, t > 0. \tag{1.3}$$

Proof. The first step is to establish the identity in the special case when $t = 1/2$ and the dimension is $n = 1$. Set $u = e^{\frac{-x^2}{2}}$ and first note that

$$\partial_x(u) + xu = 0. \tag{1.4}$$

On the other hand, using (1.1), (1.2), and (1.4) we see that

$$\partial_\xi(\mathcal{F}(u)) + \xi\mathcal{F}(u) = 0. \tag{1.5}$$

By the uniqueness of solutions to ODEs, this implies that $\mathcal{F}(u)(\xi) = Cu(\xi)$ for some constant C. Using the identity

$$\mathcal{F}(u)(0) = \int_{\mathbb{R}} e^{\frac{-x^2}{2}}\,\mathrm{d}x = \sqrt{2\pi}$$

we see that $C = \sqrt{2\pi}$.

Now, for the general $n = 1$ case we make the change of variables $x' = \sqrt{2t}x$ to obtain

$$\int_{\mathbb{R}} e^{-ix\xi} e^{-tx^2}\,\mathrm{d}x = \int_{\mathbb{R}} e^{-ix\xi/\sqrt{2t}} \frac{e^{\frac{(x')^2}{2}}}{\sqrt{2t}}\,\mathrm{d}x = \frac{1}{\sqrt{2t}}\mathcal{F}(e^{-x^2/2})(\xi/\sqrt{2t}).$$

Using the previous calculation now proves the result in the case $n = 1$. By Fubini's theorem we may reduce the n dimensional case to the one dimensional case and thus the result is proven. □

We now introduce the inverse Fourier Transform. For $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\mathcal{F}^{-1}(u)(\xi) = \frac{1}{(2\pi)^n}\mathcal{F}(u)(-\xi).$$

The next task is to prove that indeed this operator, $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, warrants the title inverse Fourier Transform.

Theorem 1.3 (Fourier Inversion Theorem). *We have*

$$\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}.$$

Proof. Take any $u \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
[\mathcal{F}^{-1}\mathcal{F}u](y) &= \frac{1}{(2\pi)^n}\int e^{iy\cdot\xi}\left(\int e^{-ix\cdot\xi}u(x)\,\mathrm{d}x\right)\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\lim_{\epsilon\to 0^+}\int e^{iy\cdot\xi}e^{-\epsilon|\xi|^2}\left(\int e^{-ix\cdot\xi}u(x)\,\mathrm{d}x\right)\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\lim_{\epsilon\to 0^+}\int u(x)\left(\int e^{i(y-x)\cdot\xi}e^{-\epsilon|\xi|^2}\,\mathrm{d}\xi\right)\mathrm{d}x \\
&= \frac{1}{2^n\pi^{n/2}}\lim_{\epsilon\to 0^+}\int u(x)\frac{e^{-\frac{|x-y|^2}{4\epsilon}}}{\epsilon^{n/2}}\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^{n/2}}\lim_{\epsilon\to 0^+}\int u(y+x\sqrt{2\epsilon})e^{-\frac{|x|^2}{2}}\,\mathrm{d}x \\
&= u(y).
\end{aligned}$$

Therefore $\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}$, and $\mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}$ follows almost immediately. □

We will sometimes use the notation

$$\check{u} = \mathcal{F}^{-1}(u).$$

The next theorem shows that it might have been smarter to include a factor of $(2\pi)^{-n/2}$ in the definition of $\mathcal{F}$ since then $\mathcal{F}$ would be a unitary operator on $L^2(\mathbb{R}^n)$.

Theorem 1.4. *For u, $v \in \mathcal{S}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n}\hat{u}\,\bar{v}\,\mathrm{d}x = (2\pi)^n\int_{\mathbb{R}^n}u\,\bar{\check{v}}\,\mathrm{d}x.$$

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{u}(x)\,\bar{v}(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(\xi)\,\bar{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(\xi)\,\overline{e^{ix\cdot\xi}v(x)}\,\mathrm{d}\xi\,\mathrm{d}x. \\ &= (2\pi)^n \int_{\mathbb{R}^n} u(\xi)\,\bar{\check{v}}(\xi)\,\mathrm{d}\xi.\end{aligned}$$

□

Now let $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ denote the L^2 inner product. Combining Theorems 1.3 and 1.4 we can immediately obtain the following facts for u and $v \in \mathcal{S}(\mathbb{R}^n)$. The first is just a restatement of Theorem 1.4.

$$\langle \hat{u}, v\rangle_{L^2(\mathbb{R}^n)} = (2\pi)^n \langle u, \check{v}\rangle_{L^2(\mathbb{R}^n)}. \tag{1.6}$$

$$(2\pi)^n \langle u, v\rangle_{L^2(\mathbb{R}^n)} = \langle \hat{u}, \hat{v}\rangle_{L^2(\mathbb{R}^n)}. \tag{1.7}$$

$$(2\pi)^n \|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{1.8}$$

The last of these shows, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, that $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to continuous linear operators on $L^2(\mathbb{R}^n)$.

Finally we consider how the Fourier transform behaves with respect to the L^1 norm.

Lemma 1.5. *We have the following estimates valid for* $u \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\|\hat{u}\|_{L^\infty(\mathbb{R}^n)} &\le \|u\|_{L^1(\mathbb{R}^n)}, \\ \|u\|_{L^\infty(\mathbb{R}^n)} &\le (2\pi)^{-n}\|\hat{u}\|_{L^1(\mathbb{R}^n)}, \\ \|\hat{u}\|_{L^1(\mathbb{R}^n)} &\le C \max_{|\alpha|\le n+1} \|\partial_x^\alpha u\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

The constant C only depends on the dimension n.

Proof. The first and second estimates are immediate from the definition of the Fourier transform, and the Fourier inversion formula. For the last one we have

$$\begin{aligned}\|\hat{u}\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|)^{n+1}|\hat{u}(\xi)|\frac{\mathrm{d}\xi}{(1+|\xi|)^{n+1}} \\ &\le C\sup_\xi (1+|\xi|)^{n+1}|\hat{u}(\xi)| \\ &\le C \max_{|\alpha|\le n+1}\|\xi^\alpha \hat{u}(\xi)\|_{L^\infty(\mathbb{R}^n)} \\ &\le C \max_{|\alpha|\le n+1}\|\mathcal{F}[D_x^\alpha u](\xi)\|_{L^\infty(\mathbb{R}^n)} \\ &\le C \max_{|\alpha|\le n+1}\|\partial_x^\alpha u\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

□

1.2 Extension of Fourier transform to $\mathcal{S}'$

Now we will show how to extend the Fourier transform to spaces of distributions. In particular we can extend to the so-called tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ which are the space of continuous linear functionals on the Frechet space $\mathcal{S}(\mathbb{R}^n)$ defined above. A linear map $f : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is in $\mathcal{S}'(\mathbb{R}^n)$ if and only if there exists an $M \in \mathbb{N}$, and constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$|f(u)| \leq C \sum_{|\beta|\leq M, |\alpha|\leq M} \|u\|_{\beta,\alpha}.$$

If $\{f_j\} \subset \mathcal{S}'(\mathbb{R}^n)$ is a sequence, then we say that f_j converges to f in $\mathcal{S}'(\mathbb{R}^n)$ if $f_j(u) \to f(u)$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

We can now define the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ by duality. Indeed, for $f \in \mathcal{S}'(\mathbb{R}^n)$ we define $\mathcal{F}(f)$ by

$$[\mathcal{F}(f)](u) = f(\mathcal{F}(u)).$$

Since the Fourier transform is continuous on $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}(f) \in \mathcal{S}'(\mathbb{R}^n)$. The inverse Fourier transform is defined in the similar way for $f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$[\mathcal{F}^{-1}(f)](u) = f(\mathcal{F}^{-1}(u)),$$

and it is easy to see that $\mathcal{F}^{-1}$ is still the inverse of $\mathcal{F}$. Also, it follows immediately that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are sequentially continuous on $\mathcal{S}'(\mathbb{R}^n)$ in the sense that if $f_j \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ then

$$\mathcal{F}(f_j) \to \mathcal{F}(f) \quad \text{and} \quad \mathcal{F}^{-1}(f_j) \to \mathcal{F}^{-1}(f).$$

As we will see in Lemma 1.6 at the end of this section we can in some cases use this fact to compute Fourier transforms by explicitly finding a sequence that converges in $\mathcal{S}'$ to f and whose Fourier transforms are known.

If f is any function such that $(1 + |x|)^N f(x)$ is bounded for some $N \in \mathbb{N}$, then f defines an element $T_f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$T_f(u) = \int_{\mathbb{R}^n} f(x)\, u(x)\, \mathrm{d}x.$$

In this case we will always identify f with T_f without comment. With this identification, and the fact that $\mathcal{F}(T_f)(u) = T_{\mathcal{F}(f)}(u)$, which can be proven by an argument almost identical to the proof of theorem 1.4, we see that this definition of $\mathcal{F}$ extends the definition on $\mathcal{S}(\mathbb{R}^n)$.

To finish the section we calculate a nontrivial Fourier transform for a tempered distribution which will be very useful later.

Lemma 1.6. *Let $A \in M_n(\mathbb{R}^n)$ be symmetric and non-degenerate (i.e. $|A| := \det(A) \neq 0$) with signature* $\mathrm{sgn(A)}$. *Then*

$$\mathcal{F}(e^{i(x^t A x)}) = \pi^{n/2} e^{\mathrm{sgn(A)}\frac{\pi i}{4}} \frac{e^{-\frac{i}{4}\xi^t (A^{-1})\xi}}{\sqrt{|A|}}$$

Note $e^{i(x^t Ax)}$ is bounded and so defines a tempered distribution.

Proof. This proof essentially follows from the analytic continuation of (1.3). Indeed, note that both sides of (1.3) define an analytic function on the domain $\{\mathrm{Re}(t) > 0\}$, and so by the uniqueness of analytic continuation the equality can be extended from the positive real line to the entire right half of the complex plane. Applying this fact in the case $n = 1$ we have the formula

$$\int_{\mathbb{R}} e^{-(\epsilon - ir)x^2} e^{-ix\xi}\, \mathrm{d}x = \sqrt{\pi}\frac{e^{-\frac{\xi^2}{4(\epsilon - ir)}}}{\sqrt{(\epsilon - ir)}}$$

for any $\epsilon > 0$ and $r \in \mathbb{R}$. Note that there is no problem extending the square root analytically in the right half plane.

Now observe that

$$\lim_{\epsilon \to 0^+} e^{i(x^t Ax) - \epsilon |x|^2} = e^{i(x^t Ax)}$$

in $\mathcal{S}'(\mathbb{R}^n)$, and so by the continuity of $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^n)$ we have that

$$\mathcal{F}(e^{i(x^t Ax)}) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} e^{i(x^t Ax - x\cdot\xi) - \epsilon |x|^2} \mathrm{d}x.$$

Since A is symmetric it is diagonalizable over $\mathbb{R}$, and since it is non-degenerate none of the eigenvalues are zero. Suppose the eigenvalues are $\{r_j\}_{j=1}^n$. Then if we change variables by $x \mapsto Ox$ for an appropriate $O \in \mathcal{O}(\mathbb{R}^n)$, we obtain from the last formula

$$\begin{aligned}
\mathcal{F}(e^{i(x^t Ax)}) &= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} e^{\left(\sum_{j=1}^n -(\epsilon - ir_j)(x^j)^2\right)} e^{-i(x\cdot(O^{-1}\xi))}\, \mathrm{d}x \\
&= \lim_{\epsilon \to 0^+} \prod_{j=1}^n \int_{\mathbb{R}} e^{-(\epsilon - ir_j)x^2} e^{-i(x\,(O^{-1}\xi)_j)}\, \mathrm{d}x \\
&= \lim_{\epsilon \to 0^+} \pi^{n/2} \prod_{j=1}^n \frac{e^{-\frac{(O^{-1}\xi)_j^2}{4(\epsilon - ir_j)}}}{\sqrt{(\epsilon - ir_j)}} \\
&= \pi^{n/2} \prod_{j=1}^n \frac{e^{-i\frac{(O^{-1}\xi)_j^2}{4r^j}}}{\sqrt{-ir_j}} \\
&= \pi^{n/2} e^{\mathrm{sgn(A)}\frac{\pi \mathrm{i}}{4}} \frac{e^{-\frac{i}{4}\xi^t(A^{-1})\xi}}{\sqrt{|A|}}.
\end{aligned}$$

□

1.3 Oscillatory integrals and their regularization

Oscillatory integral are a class of distributions that are defined in particular way by formal integrals that appear to be divergent. An instructive example is

$$\frac{1}{(2\pi)^n} \int e^{i\xi\cdot x} \mathrm{d}\xi. \tag{1.9}$$

How are we to make sense of this integral? Let us attempt to interpret it as a distribution, and test it against a Schwartz function u. Doing this we obtain, using the Fourier inversion theorem,

$$\frac{1}{(2\pi)^n}\iint e^{i\xi\cdot x}u(x)\,\mathrm{d}x\,\mathrm{d}\xi = \frac{1}{(2\pi)^n}\int \widehat{u}(\xi)\,\mathrm{d}\xi = u(0).$$

Thus we see that (1.9) is in fact equal to the delta distribution δ. Indeed, in the language of oscillatory integrals, the Fourier inversion theorem may be written simply as

$$\frac{1}{(2\pi)^n}\int e^{i\xi\cdot x}\mathrm{d}\xi = \delta(x).$$

Another way to interpret (1.9) is by taking a limit in the sense of distributions. This idea leads to, using Theorem 1.2

$$\begin{aligned}\frac{1}{(2\pi)^n}\int e^{i\xi\cdot x}\mathrm{d}\xi &= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}e^{-\epsilon|\xi|^2}\mathrm{d}\xi \\ &= \lim_{\epsilon\to 0^+}\frac{1}{2^n(\pi\epsilon)^{n/2}}e^{-\frac{|x|^2}{4\epsilon}} \\ &= \delta(x)\end{aligned} \tag{1.10}$$

This last method generalizes quite a bit and we will now continue to show how we may use it to interpret formal expressions similar to (1.9) as distributions.

Let X be an open subset of $\mathbb{R}^n$, and ϕ and $a \in C^\infty(X\times\mathbb{R}^N)$. Formally an oscillatory integral is defined in the following way

$$A = \int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)\,\mathrm{d}\xi. \tag{1.11}$$

If $|a(x,\xi)| \le (1+|\xi|)^\mu$ for $\mu < -n$, then in fact the integral converges absolutely. In this case A is a continuous function on X and therefore corresponds to an element of $\mathcal{D}'(X)$. Further hypotheses on a and ϕ can guarantee that A is in $C^k(X)$, or in fact $C^\infty(X)$, but we will not formulate these. Instead in the next section we describe standard assumptions on a and ϕ that allow us to interpret A as a distribution on X.

1.3.1 Phase functions, symbols, and conic sets

We will generally refer to the function a in (1.11) as the symbol and the function ϕ as the phase function of the oscillatory integral a. For a we introduce the following *symbol spaces*.

Definition 1.7. *Let μ, ρ and $\delta \in \mathbb{R}$. Then the space $S^\mu_{\rho,\delta}(X\times\mathbb{R}^N)$ is the set of those $a \in C^\infty(X\times\mathbb{R}^N)$ such that for any compact set $K \Subset X$ and multi-indices α and β there exists a constant $C_{K,\alpha,\beta}$ such that for all $x \in K$*

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \le C_{K,\alpha,\beta}(1+|\xi|)^{\mu-|\beta|\rho+|\alpha|\delta}. \tag{1.12}$$

The optimal values of the constants $C_{K,\alpha,\beta}$ provide a set of semi-norms on $S^\mu_{\rho,\delta}(X\times\mathbb{R}^N)$ which turn $S^\mu_{\rho,\delta}(X\times\mathbb{R}^N)$ into a Frechet space.

We also write

$$S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_{\mu \in \mathbb{R}} S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$$

(note that this intersection is the same for any ρ and δ) and

$$S^{\infty}_{\rho,\delta}(X \times \mathbb{R}^N) = \bigcup_{\mu \in \mathbb{R}} S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N).$$

There are a few straightforward results concerning the symbol spaces that ought to be mentioned.

Lemma 1.8. *Properties of symbols*

1. *If $a \in S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$ and $b \in S^{\mu'}_{\rho,\delta}(X \times \mathbb{R}^N)$, then $ab \in S^{\mu+\mu'}_{\rho,\delta}(X \times \mathbb{R}^N)$. Furthermore, the bilinear map $(a,b) \mapsto ab$ is continuous in both its entries.*

2. *If $a \in S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$ then $\partial_x^{\alpha}\partial_{\xi}^{\beta} a \in S^{\mu-|\beta|\rho+|\alpha|\delta}_{\rho,\delta}(X \times \mathbb{R}^N)$ and the map $a \mapsto \partial_x^{\alpha}\partial_{\xi}^{\beta} a$ is continuous between these spaces.*

We leave the proof of this lemma as an exercise.

Example 1.9 (Examples of symbols) Suppose that $\phi \in C_c^{\infty}(\mathbb{R}^N)$ and $\psi \in C^{\infty}(X)$. Of course then $\psi(x)\phi(\xi) \in S^{-\infty}_{\rho,\delta}(X \times \mathbb{R}^N)$ for any ρ and δ. If ϕ is equal to one on a neighborhood of the origin and $a \in C^{\infty}(\mathbb{R}^N \setminus \{0\})$ is positive homogeneous of order μ (i.e. for any $\lambda > 0$ and $\xi \neq 0$, $a(\lambda\xi) = \lambda^{\mu} a(\xi)$), then

$$\psi(x)(1-\phi(\xi))a(\xi) \in S^{\mu}_{1,0}(X \times \mathbb{R}^N)$$

and

$$\psi(x)(1-\phi(\xi))a(\xi)e^{i|x|^2|\xi|^{\delta}} \in S^{\mu}_{1-\delta,\delta}(X \times \mathbb{R}^N).$$

∎

In some cases it is necessary to consider functions that only satisfy the requirements in definition 1.7 on a set of the form $X \times \{\xi \in \mathbb{R}^N \ : \ |\xi| > R\}$ for some $R \in \mathbb{R}$. We say that such functions are symbols for R sufficiently large.

The following definition gives the assumptions we will make on ϕ.

Definition 1.10. *A real-valued phase function on $X \times \mathbb{R}^N$ is a real-valued function $\phi(x,\xi) \in C^{\infty}(X \times (\mathbb{R}^N \setminus \{0\})$ that is positive homogeneous of degree 1 in ξ, and has no critical points.*

As we will see in the next section when $a \in S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$ for some $\rho > 0$ and $\delta < 1$ and ϕ is a real-valued phased function on $X \times \mathbb{R}^N$ then (1.11) defines a distribution. It is worthwhile to comment that more general or just different hypothesis on a and ϕ may also work, but the definitions given here cover all applications in which we are interested.

1.3.2 Regularization

With these definitions we now have the following theorem which shows how the right hand side of (1.11) may be interpreted as a distribution. Equation (1.13) is the true definition of an oscillatory integral, and should be compared with (1.10) where $\chi(\xi) = e^{-|\xi|^2}$ which does not quite fit the hypotheses of the theorem but nonetheless works the same way in the case of (1.10).

Theorem 1.11. *Suppose that ϕ is a real-valued phase function on $X \times \mathbb{R}^N$ and $a \in S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$ for some $\rho > 0$, and $\delta < 1$. Also let $\chi \in \mathcal{S}(\mathbb{R}^N)$ have $\chi = 1$ on a neighborhood of the origin, and $u \in C_0^\infty(X)$. Then the limit*

$$A_{\phi,a}(u) = \lim_{\epsilon \to 0+} \int_{X \times \mathbb{R}^N} e^{i\,\phi(x,\xi)} a(x,\xi)\chi(\epsilon\xi)u(x)\,\mathrm{d}x\,\mathrm{d}\xi \tag{1.13}$$

exists and is independent from χ. For any integer $k > (\mu + n)/\min(1-\delta, \rho)$ the map $u \mapsto A_{\phi,a}(u)$ defines a distribution on X of order k. Further, the map $a \mapsto A_{\phi,a}(u)$ is continuous on the Frechet space $S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$.

Proof. Choose a cutoff function $\psi \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(\psi) \subset B_2(0)$ and $\psi(\xi) = 1$ on $B_1(0)$ and consider the differential operator

(1.14)

$$L = \psi(\xi) + \frac{(1-\psi(\xi))}{i(|\partial_x\phi(x,\xi)|^2 + |\xi|^2|\partial_\xi\phi(x,\xi)|^2)} \left(\sum_j \partial_{x^j}\phi\,\partial_{x^j} + |\xi|^2 \sum_j \partial_{\xi^j}\phi\,\partial_{\xi^j} \right).$$

Then we have $Le^{i\phi(x,\xi)} = e^{i\phi(x,\xi)}$, and by the hypotheses on ϕ

$$L^t = A(x,\xi) + \sum_j B_j(x,\xi)\partial_{x^j} + \sum_j C_j(x,\xi)\partial_{\xi^j}$$

where A and each B_j is in $S^{-1}_{1,0}(X \times \mathbb{R}^N)$, and the C_j are in $S^0_{1,0}(X \times \mathbb{R}^N)$. Therefore

$$(L^t)^k a(x,\xi)u(x) \in S^{\mu - k\min(1-\delta,\rho)}_{\rho,\delta}(X \times \mathbb{R}^N).$$

Now we claim that

$$(L^t)^k a(x,\xi)\chi(\epsilon\xi)u(x) = \chi(\epsilon\xi)(L^t)^k a(x,\xi)u(x) + \sum_{j=1}^{k} \epsilon^j D_j(x,\xi,\epsilon)$$

where each D_j is a sum of terms each having the form $d(\epsilon\xi)\,f(x,\xi)$ with each $d \in C_c^\infty(\mathbb{R}^N)$ having $\operatorname{supp}(d) \subset B_2(0) \setminus B_1(0)$, and $e \in S^{\mu-(k-j)\min(1-\delta,\rho)}_{\rho,\delta}(X \times \mathbb{R}^N)$ with $\operatorname{supp}(f(\cdot,\xi)) \subset \operatorname{supp}(u)$ for every ξ. This claim may be established by induction. Indeed, suppose the claim is true for some k. Then

$$\begin{aligned}(L^t)^{k+1} a(x,\xi)\chi(\epsilon\xi)u(x) &= \chi(\epsilon\xi)(L^t)^{k+1}a(x,\xi)u(x) \\ &\quad + \epsilon\big((L^t)^k a(x,\xi)u(x)\big)\sum_l C_l(x,\xi)\partial_{\xi^l}\chi(\epsilon\xi) + \sum_{j=1}^{k}\epsilon^j L^t D_j(x,\xi,\epsilon).\end{aligned}$$

The first sum in the previous formula consists of terms in the correct form for D_1. On the the other hand, $L^t D_j$ is a sum of terms of the form

$$d(\epsilon\xi)\, L^t f(x,\xi) + \epsilon f(x,\xi) \sum_l C_l(x,\xi)\partial_{\xi^l} d(\epsilon\xi)$$

which will produce terms in the correct form for D_j and D_{j+1}. This proves the claim.

Now choose an integer $k > (\mu + N)/\min(1-\delta,\rho)$. Then we have

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{X\times\mathbb{R}^N} & e^{i\,\phi(x,\xi)} a(x,\xi)\chi(\epsilon\xi)u(x)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \lim_{\epsilon\to 0} \int_{X\times\mathbb{R}^N} e^{i\,\phi(x,\xi)} (L^t)^k a(x,\xi)\chi(\epsilon\xi)u(x)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \lim_{\epsilon\to 0} \int_{X\times\mathbb{R}^N} e^{i\,\phi(x,\xi)} \left(\chi(\epsilon\xi)(L^t)^k a(x,\xi)u(x) + \sum_{j=1}^{k} \epsilon^j D_j(x,\xi,\epsilon)\right)\mathrm{d}x\,\mathrm{d}\xi
\end{aligned}$$

Since $(L^t)^k a(x,\xi)\,u(x) \in S^{-N-\alpha}_{\rho,\delta}(X\times\mathbb{R}^N)$ for some small $\alpha > 0$, by the dominated convergence theorem

$$\lim_{\epsilon\to 0} \int_{X\times\mathbb{R}^N} e^{i\,\phi(x,\xi)}\chi(\epsilon\xi)(L^t)^k a(x,\xi)\,u(x)\mathrm{d}x\,\mathrm{d}\xi = \int_{X\times\mathbb{R}^N} e^{i\,\phi(x,\xi)}(L^t)^k a(x,\xi)u(x)\,\mathrm{d}x\,\mathrm{d}\xi.$$

For each of the terms $d(\epsilon\xi)\,f(x,\xi)$ making up D_j we have

$$\begin{aligned}
\left|\epsilon^j \int_{X\times\mathbb{R}^N} e^{i\,\phi(x,\xi)} d(\epsilon\xi)\,f(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\right| &\lesssim \epsilon^{j-n} \sup_{x\in\mathrm{supp}(u),\,1/\epsilon\le|\xi|\le 2/\epsilon} |f(x,\xi)| \\
&\lesssim \epsilon^{\alpha+j(1-\min(1-\delta,\rho))}
\end{aligned}$$

for some small $\alpha > 0$. Therefore the limit does indeed exists as claimed in the theorem and

$$A = \int_{X\times\mathbb{R}^N} e^{i\,\phi(x,\xi)}(L^t)^k a(x,\xi)u(x)\,\mathrm{d}x\,\mathrm{d}\xi. \tag{1.15}$$

Thus $|A|$ can be estimated by the derivatives of u up to order k and some finite number of the $S^{\mu}_{\rho,\delta}(X\times\mathbb{R}^N)$ semi-norms of a, which completes the proof.
□

For the record, we now state the definition of an oscillatory integral.

Definition 1.12. *An oscillatory integral is a distribution on an open set $X \subset \mathbb{R}^n$ defined by a real-valued phase function ϕ on X and a symbol $a \in S^{\infty}_{\rho,\delta}(X\times\mathbb{R}^N\setminus\{0\})$ for some $\rho > 0$ and $\delta < 1$ as in (1.13). We will often write oscillatory integrals in the formal form (1.11).*

It is worth noting that for any oscillatory integral the proof of the previous theorem shows $\langle A, u\rangle$ can be given by a formula of the form (1.15) in which the integral converges absolutely.

Such a formula is called a *regularization* of the oscillatory integral A. For a given oscillatory integral there are many possible regularizations depending on all of the various choices made in the proof of Theorem 1.11 but they all give the same result.

We may slightly generalize Theorem 1.11 to apply for symbols a that are not strictly in $S^{\mu}_{\rho,\delta}(X \times \mathbb{R}^N)$ with only a small change in the proof. Indeed, a may have integrable singularities as a function of ξ contained in a compact subset of $\mathbb{R}^N$ and the only change required in the proof is that the cutoff function ψ should be equal to 1 on the set where the singularities lie (rather than just on $B_1(0)$). In this case the hypotheses change to ask that $a(x, \xi)$ must satisfy the requirements of definition 1.12 for $|\xi|$ sufficiently large.

We may also find formulas for the derivatives of oscillatory integrals. Indeed, since differentiation is continuous on the space of distributions we conclude that

$$\langle \partial_x A, u \rangle = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \chi(\epsilon\xi) e^{i\phi(x,\xi)} \left(i\, \partial_x \phi(x, \xi)\, a(x, \xi) + \partial_x a(x, \xi)\right) u(x)\, \mathrm{d}x\, \mathrm{d}\xi$$

for any function χ as in the theorem. Note that the symbol in this integral is not differentiable at $\theta = 0$ (ϕ is not assumed to be differentiable at $\xi = 0$), but still satisfies the requirements of definition 1.12 for $|\xi| > 1$ with μ incremented by 1. Higher derivatives may be computed in the same way. Further, if ϕ and a depend on a parameter, then the distribution A also depends on the parameter with the same level of regularity as ϕ and a, and we may always differentiate “under the integral sign" in oscillatory integrals.

1.4 Exercises

1. The space $\mathcal{E}'(\mathbb{R}^n)$ of compactly supported distributions is contained in $\mathcal{S}'(\mathbb{R}^n)$. Show that if $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\mathcal{F}[u] \in C^\infty(\mathbb{R}^n)$ is given by

$$\mathcal{F}[u](\xi) = \langle u, e^{-ix\cdot\xi} \rangle.$$

   Furthermore, prove that there exists a integer N and constant C such that

$$|\mathcal{F}[u](\xi)| \le C(1 + |\xi|)^N.$$

2. Let $\mathrm{sgn}(x) \in \mathcal{S}'(\mathbb{R})$ be the sign function

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0. \end{cases}$$

   Show that

$$\mathcal{F}[\mathrm{sgn}] = -2\,\mathrm{i}\, \left(p.v. \frac{1}{\xi}\right)$$

   where $p.v. \frac{1}{\xi} \in \mathcal{S}'(\mathbb{R})$ is the principal value distribution defined by

$$\left\langle p.v. \frac{1}{\xi}, \varphi \right\rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{\varphi(\xi)}{\xi}\, \mathrm{d}\xi.$$

   (Hint: $\lim_{\epsilon\to 0^+} \mathrm{sgn}(x) e^{-\epsilon|x|} = \mathrm{sgn}(x)$ where the limit is in the sense of distributions.)

3. Let $H(x)$ be the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Show that

$$\mathcal{F}[H] = -\mathrm{i}\left(p.v.\frac{1}{\xi}\right) + \pi\delta.$$

4. Suppose that $f \in C^\infty(\mathbb{S}^{n-1})$ and k is an integer such that $-n < k < 0$ and

$$f(\omega) = (-1)^{k+n-1} f(-\omega).$$

Then

$$g(x) = f\left(\frac{x}{|x|}\right)|x|^k \in L^1_{loc}(\mathbb{R}^n)$$

defines a tempered distribution. In this exercise we will calculate $\mathcal{F}[g]$ in several steps.

(a) Show that

$$\mathcal{F}[g] = \lim_{\epsilon \to 0^+} \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} e^{-\mathrm{i}r\omega\cdot\xi} e^{-\epsilon|r|} f(\omega) r^{k+n-1}\, \mathrm{d}r\, \mathrm{d}H_{\mathbb{S}^{n-1}}(\omega).$$

(b) Using the previous part show that

$$\langle \mathcal{F}[g], \varphi\rangle = \pi\, \mathrm{i}^{k+n-1} \langle \delta^{(n+k-1)}(\omega\cdot\xi), f(\omega)\varphi(\xi)\rangle.$$

(c) Finally show

$$\mathcal{F}[g](\xi) = \frac{\pi|\xi|^{-n-k}}{\mathrm{i}^{k+n-1}} \int_{\{\omega\cdot\xi=0\}} \left(\frac{\xi}{|\xi|}\cdot\nabla_\omega\right)^{n+k-1} f(\omega)\, \mathrm{d}H_{\{\omega\cdot\xi=0\}}(\omega).$$

5. Prove lemma 1.8.