Convergence of the Secant Method CAAM 553: Numerical Analysis I

Let f be a real valued nonlinear function of a real variable. Assume throughout that x_* is a real number such that $f(x_*) = 0$ and that $f'(x_*) \neq 0$ with $f''(x)$ continuous in some open interval containing x_*.

Lemma 1 *Let $0 < c < 1$ be given. Then there is an interval $N_\delta = \{x : |x - x_*| < \delta\}$ such that*

$$\left| \left(\frac{\xi - \eta}{f(\xi) - f(\eta)} \right) \left[f'(\zeta) - \frac{f(\xi) - f(\eta)}{\xi - \eta} \right] \right| < c$$

for all $\xi, \eta, \zeta \in N_\delta$.

Proof. As $\delta \to 0$ we have $\left(\frac{f(\xi)-f(\eta)}{\xi-\eta}\right) \to f'(x_*) \neq 0$ and $f'(\zeta) \to f'(x_*)$. This follows from the definition of derivative, the fact that $\xi, \eta, \zeta \to x_*$ and from the continuity of $f'(x)$. The result is a direct consequence. □
(Providing a detailed proof is left as an exercise)

Now, given x_0, x_1, let the sequence $\{x_k\}$ result from the Secant Method with

$$x_{k+1} = x_k - \left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \right) f(x_k), \quad \text{for } k = 2, 3, 4, \ldots.$$

Without loss of generality we assume $x_k - x_* \neq 0$ for any k for otherwise the iteration would have been halted.

For the remainder of this discussion we shall introduce the notation $f_k = f(x_k), f'_k = f'(x_k), f''_k = f''(x_k)$ and $f_* = f(x_*), f'_* = f'(x_*), f''_* = f''(x_*)$ with $e_k = x_k - x_*$.

Lemma 2 *Let $0 < c < 1$ be given and let N_δ be defined as in Lemma 1. If $x_0, x_1 \in N_\delta$ then $x_k \in N_\delta$ for $k = 2, 3, 4, \ldots$ with*

$$\lim_{k \to \infty} x_k = x_* \quad \text{and} \quad \lim_{k \to \infty} \frac{e_k}{e_{k-1}} = 0.$$

Proof. Taylor's formula with remainder (expanded about x_k) gives

$$0 = f_* = f_k - f'(\xi_k) e_k = f_k - \left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right) e_k - \left[f'(\xi_k) - \left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right) \right] e_k$$

with ξ_k between x_k and x_*. Solving for $e_k = x_k - x_*$ gives

$$x_k - x_* = \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right) f_k - \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right) \left[f'(\xi_k) - \left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right) \right] e_k$$

Hence, from the definition of x_{k+1},

$$e_{k+1} = x_{k+1} - x_* = -\left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right)\left[f'(\xi_k) - \left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}}\right)\right] e_k. \tag{1}$$

Therefore, if $x_{k-1}, x_k \in \mathrm{N}_\delta$ then Lemma 1 implies that

$$|e_{k+1}| \leq c|e_k|.$$

Now, if $x_0, x_1 \in \mathrm{N}_\delta$ the above argument implies that $|e_2| < c|e_1| < \delta$ since $c < 1$ so that $x_2 \in \mathrm{N}_\delta$. Successively repeating this provides $x_k \in \mathrm{N}_\delta$ for all k and that $|e_k| < c|e_{k-1}|$ for $k \geq 2$. Observe that

$$|e_{k+1}| < c|e_k| < c^2|e_{k-1}| < \cdots < c^k e_1, \ \text{ for } \ k > 1$$

and $x_k \to x_*$ follows since $0 < c < 1$.

Finally, $x_k \to x_*$ together with Equation 1 implies

$$\lim_{k\to\infty} \frac{|e_k|}{|e_{k-1}|} = 0 \tag{2}$$

to conclude the proof. □

With these results, defining $\hat{\gamma}_k = \frac{|e_k|}{|e_{k-1}|}$ will provide the relationship

$$|e_k| = \hat{\gamma}_k |e_{k-1}| \ \text{ for } k = 2, 3, 4, \ldots$$

with $\hat{\gamma}_k \to 0$. Thus it is now reasonable to ask for the largest positive exponent $\eta > 0$ such that $\gamma_k \equiv \hat{\gamma}_k / |e_k|^\eta \leq \bar{\gamma}$ is bounded. Hence we seek a power law relationship

$$|e_{k+1}| = \gamma_k |e_k|^\alpha \tag{3}$$

with $\alpha = 1 + \eta$. To this end, we first intend to show that

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A \ \text{ with } \ A \equiv \frac{f''_*}{2f'_*}$$

and then to show that this result together with the assumption of the power law (Equation 3) will imply $\alpha = (1 + \sqrt{5})/2$.

Lemma 3 *Suppose that the Secant iteration x_k converges to x_* in accordance with Lemma 1. Then*

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A \ \text{ with } \ A \equiv \frac{f''_*}{2f'_*}. \tag{4}$$

Proof. First note that the Secant iteration can be written in the form

$$x_{k+1} = \frac{f_k x_{k-1} - f_{k-1} x_k}{f_k - f_{k-1}}.$$

Hence,

$$e_{k+1} = x_{k+1} - x_* = \frac{f_k x_{k-1} - f_{k-1} x_k}{f_k - f_{k-1}} - x_* = \frac{f_k e_{k-1} - f_{k-1} e_k}{f_k - f_{k-1}}. \tag{5}$$

Now, use the Taylor expansions

$$f_k = f_* + f'_* e_k + \frac{1}{2} f''(\xi_k) \ \text{ and } \ f_{k-1} = f_* + f'_* e_{k-1} + \frac{1}{2} f''(\xi_{k-1})$$

together with the fact that $f_* = 0$ to obtain

$$\begin{aligned} f_k e_{k-1} - f_{k-1} e_k &= f'_*(e_{k-1} e_k - e_k e_{k-1}) + \frac{1}{2}[f''(\xi_k) e_k^2 e_{k-1} - f''(\xi_{k-1}) e_{k-1}^2 e_k] \\ &= \frac{1}{2}[f''(\xi_k) e_k - f''(\xi_{k-1}) e_{k-1}] e_k e_{k-1} \\ &= \frac{1}{2} f''_*(x_k - x_{k-1}) + \frac{1}{2}[(f''(\xi_k) - f''_*) e_k - (f''(\xi_{k-1}) - f''_*) e_{k-1}] e_k e_{k-1}. \end{aligned}$$

Now, from Equation 5 we obtain

$$\begin{aligned} e_{k+1} &= \frac{(f_k e_{k-1} - f_{k-1} e_k)/(x_k - x_{k-1})}{(f_k - f_{k-1})/(x_k - x_{k-1})} \\ &= \frac{1}{2} f''_* \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right) e_k e_{k-1} \\ &+ \frac{1}{2} \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right) \left(\frac{(f''(\xi_k) - f''_*) e_k - (f''(\xi_{k-1}) - f''_*) e_{k-1}}{x_k - x_{k-1}} \right) e_k e_{k-1}. \end{aligned} \tag{6}$$

Use $x_k - x_{k-1} = e_k - e_{k-1}$ to obtain (for k sufficiently large) that

$$\begin{aligned} \left| \frac{(f''(\xi_k) - f''_*) e_k - (f''(\xi_{k-1}) - f''_*) e_{k-1}}{x_k - x_{k-1}} \right| &= \left| \frac{(f''(\xi_k) - f''_*) \frac{e_k}{e_{k-1}} - (f''(\xi_{k-1}) - f''_*)}{\frac{e_k}{e_{k-1}} - 1} \right| \\ &\le \frac{|(f''(\xi_k) - f''_*) \frac{e_k}{e_{k-1}}| + |f''(\xi_{k-1}) - f''_*|}{1 - |\frac{e_k}{e_{k-1}}|} \\ &\le \frac{|(f''(\xi_k) - f''_*)| c + |f''(\xi_{k-1}) - f''_*|}{1 - c}, \end{aligned} \tag{7}$$

where $0 < c < 1$ is given in Lemma 1. The continuity of f'' at x_* assures that the right hand side of Inequality 7 will tend to 0 as k tends to ∞.

Since $(f_k - f_{k-1})/(x_k - x_{k-1}) \to f'_*$ as $k \to 0$, it now follows from Equation 6 that the desired limit in Equation 4 is obtained and this concludes the proof. □

To get the final result, we shall have to assume that the limit $A > 0$ which amounts to assuming $f''_* \neq 0$.

Lemma 4 *Assume that the Secant iteration x_k converges to x_* in accordance with Lemma 1 and that*

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A > 0.$$

If

$$|e_{k+1}| = \gamma_k |e_k|^\alpha \ \text{ with } \lim_{k\to\infty} \gamma_k = \gamma$$

then

$$\alpha^2 - \alpha - 1 = 0 \ \text{ and } \ \gamma = A^{1/\alpha}$$

with $\alpha = (1+\sqrt{5})/2$.

Proof. Since $|e_k| = \gamma_{k-1}|e_{k-1}|^\alpha$ we have $|e_{k-1}| = \left(\frac{|e_k|}{\gamma_{k-1}}\right)^{1/\alpha}$. It follows that

$$A = \lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = \lim_{k\to\infty} \frac{\gamma_{k-1}^{1/\alpha}\gamma_k|e_k|}{|e_k|^{1+1/\alpha}}.$$

Hence

$$\lim_{k\to\infty} |e_k|^{\alpha-1-1/\alpha} = \lim_{k\to\infty} \left(\frac{A}{\gamma_{k-1}^{1/\alpha}\gamma_k}\right).$$

Observe that if the exponent $\alpha - 1 - 1/\alpha \neq 0$, the left hand must converge to 0 and could not converge to the nonzero right hand side. Therefore, $\alpha^2 - \alpha - 1 = 0$ must hold with $\alpha = (1+\sqrt{5})/2$ being the postive root.

With this α, the left hand side must be equal to 1 always and thus

$$1 = \lim_{k\to\infty} \left(\frac{A}{\gamma_{k-1}^{1/\alpha}\gamma_k}\right) = \frac{A}{\gamma^{1+1/\alpha}}$$

and

$$\gamma^{(1+1/\alpha)} = A \ \text{ so that } \ \gamma = A^{1/(1+1/\alpha)} = A^{1/\alpha}$$

as claimed. □