Let $f$ be a real valued nonlinear function of a real variable. Assume throughout that $x_*$ is a real number such that $f(x_*) = 0$ and that $f''(x_*) \neq 0$ with $f'''(x)$ continuous in some open interval containing $x_*$. 

**Lemma 1** Let $0 < c < 1$ be given. Then there is an interval $N_\delta = \{ x : |x - x_*| < \delta \}$ such that 

$$
\left| \left( \frac{\xi - \eta}{f(\xi) - f(\eta)} \right) \left[ f''(\xi) - \frac{f(\xi) - f(\eta)}{\xi - \eta} \right] \right| < c
$$

for all $\xi, \eta, \xi \in N_\delta$.

**Proof.** As $\delta \to 0$ we have $\left( \frac{f(\xi) - f(\eta)}{\xi - \eta} \right) \to f'(x_*) \neq 0$ and $f''(\xi) \to f''(x_*)$. This follows from the definition of derivative, the fact that $\xi, \eta, \xi \to x_*$ and from the continuity of $f'(x)$. The result is a direct consequence. \(\square\)

(Providing a detailed proof is left as an exercise)

Now, given $x_0, x_1$, let the sequence $\{x_k\}$ result from the Secant Method with 

$$
x_{k+1} = x_k - \left( \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \right) f(x_k), \text{ for } k = 2, 3, 4, \ldots.
$$

Without loss of generality we assume $x_k - x_* \neq 0$ for any $k$ for otherwise the iteration would have been halted.

For the remainder of this discussion we shall introduce the notation $f_k = f(x_k), f'_k = f'(x_k), f''_k = f''(x_k)$ and $f_* = f(x_*), f'_* = f'(x_*), f''_* = f''(x_*)$ with $e_k = x_k - x_*$. 

**Lemma 2** Let $0 < c < 1$ be given and let $N_\delta$ be defined as in Lemma 1. If $x_0, x_1 \in N_\delta$ then $x_k \in N_\delta$ for $k = 2, 3, 4, \ldots$ with 

$$
\lim_{k \to \infty} x_k = x_* \text{ and } \lim_{k \to \infty} e_k = 0.
$$

**Proof.** Taylor’s formula with remainder (expanded about $x_\xi$) gives 

$$
0 = f_* = f_k - f'(\xi_k)e_k = f_k - \left( \frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right) e_k - \left[ f''(\xi_k) - \frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right] e_k
$$

with $\xi_k$ between $x_k$ and $x_*$. Solving for $e_k = x_k - x_*$ gives 

$$
x_k - x_* = \left( \frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right) f_k - \left( \frac{x_k - x_{k-1}}{f_k - f_{k-1}} \right) \left[ f''(\xi_k) - \frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right] e_k
$$
Hence, from the definition of $x_{k+1}$,

$$
e_{k+1} = x_{k+1} - x_* = -\left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right) \left[ f'(\xi_k) - \left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}}\right)\right] e_k. \tag{1}$$

Therefore, if $x_{k-1}, x_k \in N_\delta$ then Lemma 1 implies that

$$|e_{k+1}| \leq c|e_k|.$$ 

Now, if $x_0, x_1 \in N_\delta$ the above argument implies that $|e_2| < c|e_1| < \delta$ since $c < 1$ so that $x_2 \in N_\delta$. Successively repeating this provides $x_k \in N_\delta$ for all $k$ and that $|e_k| < c|e_{k-1}|$ for $k \geq 2$. Observe that

$$|e_{k+1}| < c|e_k| < c^2|e_{k-1}| < \cdots < c^k|e_1|, \text{ for } k > 1$$

and $x_k \to x_*$ follows since $0 < c < 1$.

Finally, $x_k \to x_*$ together with Equation 1 implies

$$\lim_{k \to \infty} \frac{|e_k|}{|e_{k-1}|} = 0 \tag{2}$$

to conclude the proof. □

With these results, defining $\hat{\gamma}_k = \frac{|e_k|}{|e_{k-1}|}$ will provide the relationship

$$|e_k| = \hat{\gamma}_k|e_{k-1}| \text{ for } k = 2, 3, 4, \ldots$$

with $\hat{\gamma}_k \to 0$. Thus it is now reasonable to ask for the largest positive exponent $\eta > 0$ such that $\gamma_k \equiv \hat{\gamma}_k/|e_k|^\eta \leq \bar{\gamma}$ is bounded. Hence we seek a power law relationship

$$|e_{k+1}| = \gamma_k|e_k|^\alpha \tag{3}$$

with $\alpha = 1 + \eta$. To this end, we first intend to show that

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A \text{ with } A \equiv \frac{f''_*}{2f'_*}$$

and then to show that this result together with the assumption of the power law (Equation 3 ) will imply $\alpha = (1 + \sqrt{5})/2$.

**Lemma 3** Suppose that the Secant iteration $x_k$ converges to $x_*$ in accordance with Lemma 1. Then

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A \text{ with } A \equiv \frac{f''_*}{2f'_*}. \tag{4}$$
Proof. First note that the Secant iteration can be written in the form
\[ x_{k+1} = \frac{f_k x_k - f_{k-1} x_{k-1}}{f_k - f_{k-1}}. \]
Hence,
\[ e_{k+1} = x_{k+1} - x_* = \frac{f_k x_k - f_{k-1} x_{k-1}}{f_k - f_{k-1}} - x_* = \frac{f_k e_k - f_{k-1} e_{k-1}}{f_k - f_{k-1}}. \] (5)
Now, use the Taylor expansions
\[ f_k = f_* + f'_* e_k + \frac{1}{2} f''(\xi_k) \quad \text{and} \quad f_{k-1} = f_* + f'_* e_{k-1} + \frac{1}{2} f''(\xi_{k-1}) \]
together with the fact that \( f_* = 0 \) to obtain
\[
\begin{align*}
f_k e_k &- f_{k-1} e_k = f'_*(e_k - e_{k-1}) + \frac{1}{2} [f''(\xi_k) e_k^2 - f''(\xi_{k-1}) e_{k-1}^2 e_k] \\
&= \frac{1}{2} [f''(\xi_k) e_k - f''(\xi_{k-1}) e_{k-1}] e_k e_{k-1} \\
&= \frac{1}{2} f''_*(x_k - x_{k-1}) + \frac{1}{2} [(f''(\xi_k) - f''_*) e_k - (f''(\xi_{k-1}) - f''_*) e_{k-1}] e_k e_{k-1}.
\end{align*}
\]
Now, from Equation 5 we obtain
\[
e_{k+1} = \frac{(f_k e_k - f_{k-1} e_k)/(x_k - x_{k-1})}{(f_k - f_{k-1})/(x_k - x_{k-1})} \\
= \frac{1}{2} f''_* \frac{x_k - x_{k-1}}{f_k - f_{k-1}} e_k e_{k-1} \\
+ \frac{1}{2} \frac{x_k - x_{k-1}}{f_k - f_{k-1}} \left( \frac{(f''(\xi_k) - f''_*) e_k - (f''(\xi_{k-1}) - f''_*) e_{k-1}}{x_k - x_{k-1}} \right) e_k e_{k-1}. \] (6)
Use \( x_k - x_{k-1} = e_k - e_{k-1} \) to obtain (for \( k \) sufficiently large) that
\[
\left| \frac{(f''(\xi_k) - f''_*) e_k - (f''(\xi_{k-1}) - f''_*) e_{k-1}}{x_k - x_{k-1}} \right| = \left| \frac{(f''(\xi_k) - f''_*) \frac{e_k}{e_{k-1}} - (f''(\xi_{k-1}) - f''_*)}{\frac{e_k}{e_{k-1}} - 1} \right| \\
\leq \left| \frac{(f''(\xi_k) - f''_* \frac{e_k}{e_{k-1}}) + (f''(\xi_{k-1}) - f''_*)}{1 - \frac{e_k}{e_{k-1}}} \right| \\
\leq \frac{|f''(\xi_k) - f''_*| c + |f''(\xi_{k-1}) - f''_*|}{1 - c}, \] (7)
where \( 0 < c < 1 \) is given in Lemma 1. The continuity of \( f'' \) at \( x_* \) assures that the right hand side of Inequality 7 will tend to 0 as \( k \) tends to \( \infty \).

Since \( (f_k - f_{k-1})/(x_k - x_{k-1}) \to f'_* \) as \( k \to 0 \), it now follows from Equation 6 that the desired limit in Equation 4 is obtained and this concludes the proof. □
To get the final result, we shall have to assume that the limit $A > 0$ which amounts to assuming $f'' \neq 0$.

**Lemma 4** Assume that the Secant iteration $x_k$ converges to $x_*$ in accordance with Lemma 1 and that

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = A > 0.$$ 

If

$$|e_{k+1}| = \gamma_k |e_k|^\alpha$$

with $\lim_{k \to \infty} \gamma_k = \gamma$

then

$$\alpha^2 - \alpha - 1 = 0$$

and $\gamma = A^{1/\alpha}$

with $\alpha = (1 + \sqrt{5})/2$.

**Proof.** Since $|e_k| = \gamma_{k-1} |e_{k-1}|^\alpha$ we have $|e_{k-1}| = \left(\frac{|e_k|}{\gamma_{k-1}}\right)^{1/\alpha}$. It follows that

$$A = \lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|} = \lim_{k \to \infty} \frac{\gamma_{k-1}^{1/\alpha} \gamma_k |e_k|}{|e_k|^{1+1/\alpha}}.$$

Hence

$$\lim_{k \to \infty} |e_k|^{\alpha - 1 - 1/\alpha} = \lim_{k \to \infty} \left(\frac{A}{\gamma^{1/\alpha}}\right).$$

Observe that if the exponent $\alpha - 1 - 1/\alpha \neq 0$, the left hand must converge to 0 and could not converge to the nonzero right hand side. Therefore, $\alpha^2 - \alpha - 1 = 0$ must hold with $\alpha = (1 + \sqrt{5})/2$ being the postive root.

With this $\alpha$, the left hand side must be equal to 1 always and thus

$$1 = \lim_{k \to \infty} \left(\frac{A}{\gamma_{k-1}^{1/\alpha}}\right) = \frac{A}{\gamma^{1+1/\alpha}}$$

and

$$\gamma^{(1+1/\alpha)} = A \quad \text{so that} \quad \gamma = A^{1/(1+1/\alpha)} = A^{1/\alpha}$$

as claimed. $\square$