CAAM 453: Numerical Analysis I

Problem Set 7

Due: Friday, November 21 by 9am (beginning of class).

Note: All Matlab functions mentioned in this homework assignment can be found on the CAAM453 homepage, or come with Matlab. You can use the Matlab codes posted on the CAAM453 web-page. If you modify these codes, please turn in the modified code. Otherwise you do not have to turn in printouts of the codes posted on the CAAM453 web-page. Turn in all Matlab code that you have written and turn in all output generated by your Matlab functions/scripts. Matlab functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.

Problem 1 (25 points)

The Kermack-McKendrick model for the course of an epidemic in a population is given by the system of ODEs

$$\begin{aligned} y_1' &= -cy_1y_2, \\ y_2' &= cy_1y_2 - dy_2, \\ y_3' &= dy_2 \end{aligned}$$

where $'$ denotes differentiation with respect to time, y_1 represents susceptibles, y_2 represents infectives in circulation, and y_3 represents infectives removed by isolation, death, or recovery and immunity. The parameters c and d represent the infection rate and removal rate, respectively.

- Use a Matlab library routine (of your choice) to solve this system numerically with parameter values $c = 1$ and $d = 5$, and initial values $y_1(0) = 95$, $y_2(0) = 5$, $y_3(0) = 0$. Integrate the system from $t = 0$ to $t = 1$. Plot each solution component as a function of t on the same graph using different line types.
- Experiment with other values of the parameters and initial conditions. Can you find values for which the epidemic does not grow, or for which the entire population is wiped out?

Problem 2 (35 points)

Solve the heat equation

$$u_t = \alpha u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$$

for the unknown scalar function $u(x,t)$ with initial conditions

$$u(x,0) = \sin(\pi x)$$

and boundary conditions

$$u(0,t) = 0, u(1,t) = 0.$$

You will have to set up the system of linear ODEs for the semi-discrete problem as we did in class. The trapezoidal rule will be used to integrate in time (this is commonly called a Crank-Nicholson scheme).

Please write your own code to solve this problem using the Trapezoid rule. Integrate from $t = 0$ to $t = 0.1$. Plot the computed solution, preferably as a three-dimensional surface over the (t,x) plane. Determine the maximum error in the computed solution by comparing with the exact solution

$$u(x,t) = \exp(-\pi^2 t)\sin(\pi x).$$

Set the problem up by taking

```
h = 1/(2^k + 1);
x = [ 0:h:1 ]';
```

and do runs for `k = 4,5, ..., 10` in order to do the varying mesh size experiments. Be sure to set the matrix up as a sparse matrix (you can do this either with `spdiags` or by setting up a dense `A` and then `A = sparse(A)`).

Use Matlab's backslash to solve the necessary linear systems.[1]

Please give plots of the solutions for the coarsest mesh (k = 4) and the finest mesh (k = 10).

Also solve the system using one of the built in ODE solvers in Matlab. Compare your solution with the Matlab solution (Use norms!!!).

Problem 3 (15 points)

Prove that the scheme used in Problem 2 is marginally stable for any choice of the time step. Use a Von-Neuman stability analysis as it was don in class for the forward Euler scheme.

Problem 4 (25 points) (this problem is pledged)

The Runge and Kutta RK22 method for solving

$$y' = f(y,t), \;\; y(0) = y_0$$

is given by

$$y_{k+1} = y_k + hf\left(y_k + \frac{1}{2}hf(t_k, y_k), t_k + \frac{1}{2}h\right).$$

[1] Note, for fixed stepsize you should only have one factorization per problem and then you could re-use the lu-decomposition.

- In which condition(s) the RK22 method is strictly equivalent to Heun's method?
- Use the linear ODE

$$y' = \lambda y$$

  to analyze the stability of the RK22. Graph the stability region using Matlab. Shade the interior of stablity region either by hand or with Matlab.
- Give an analysis showing that the method is order 2.