CAAM 453: Numerical Analysis I

Problem Set 4

Due: October 15th 2014 by 9am (beginning of class).

Note: All MATLAB *functions mentioned in this homework assignment can be found on the CAAM453 homepage, or come with* MATLAB . *You can use the* MATLAB *codes posted on the CAAM453 web-page. If you modify these codes, please turn in the modified code. Otherwise you do not have to turn in printouts of the codes posted on the CAAM453 web-page. Turn in all* MATLAB *code that you have written and turn in all output generated by your* MATLAB *functions/scripts.* MATLAB *functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.*

Problem 1 (20 points)

Find **by hand** the Singular Value Decomposition (SVD) of the two following matrices

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 2 (15 points)

Consider a real matrix $A \in R^{m\times n}$ of full column rank n. Show that the matrix

$$P = A(A^TA)^{-1}A^T$$

is an orthogonal projector onto $Ran(A)$.

Hint: use the SVD of A.

Problem 3 (15 points)

Consider the SVD of $A \in C^{m\times n}$:

$$A = \sum_{k=1}^{r} \sigma_k u_k v_k^*. \tag{1}$$

In class, we have shown that

$$\|A\|_2 = \sigma_1.$$

Give an expression of the Frobenius norm of A

$$\|A\|_F = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$$

as a function of the its singular values $\sigma_i,\ i = 1,\dots,r$.

Problem 4 (50 points) (this problem is pledged!)

Consider the Linear Least Squares (LLS) problem (§3.1 in Mark Embree's ebook). Find $x \in C^n$ solution of

$$\min_x \|Ax - b\|_2 \tag{2}$$

with $A \in C^{m \times n}$ and $m \geq n$. You are going to find the solution of that problem using the SVD (1) of A.

- Then, show that the solution $x^{\min}$ of (2) is

$$x^{\min} = \sum_{i=1}^{r} \frac{u_i^* b}{\sigma_i} v_i.$$

- Show that the error E is

$$E = \|Ax^{\min} - b\|_2 = \sum_{i=r+1}^{m} (u_i^* b)^2.$$

- Write a MATLAB code that solves the LLS problem using the SVD. The calling sequence should be

```
x     =   LLS_SVD(A,b)

Input:

     A   is a m x n matrix
     b   is a m vector

Output:

     x   is the solution of the LLS problem
```

  You can use the MATLAB `svd` function for computing the SVD.

- Houston's average monthly temperatures can be found in `http://www.visithoustontexas.com/travel-tools/weather/`.

  Our aim is to find a function $T(d)$ that approximates the temperature high's in Houston for any day d of a year. It is reasonable to assume that, without climate change, function T is periodic with a period of one year. Assume

$$T(d) = c_0 + c_1 \cos(\alpha d).$$

Find the right α that ensure a period of one year. Then, use the `LLS_SVD` code to find optimal values of c_0 and c_1 that fit data at best in the least square sense. Plot $T(d)$ and the datas on the same graph. Compute the error $E(c_0, c_1)$ as defined previously.

Try to find a way to improve the solution while only introducing one new parameter c_2. Quantify the improvement i.e. compute the new error $E(c_0, c_1, c_2)$ and see how the introduction of c_2 improves the approximation.

Finally, have a look on the precipitation datas on the same website and find a function $P(d)$ with 2 parameters c_0 and c_1 that approximates at best precipitations in Houston in a year. Compute $E(c_0, c_1)$.